Homework9

1.
$f(x) = (1-x)^{\alpha}$
$f'(x) = -\alpha(1-x)^{\alpha-1}, f''(x) = (-1)^2\alpha(\alpha-1)(1-x)^{\alpha-2},$
$f^{(n)}(x) = (-1)^n\alpha(\alpha-1)...(\alpha-(n-1)) = (-1)^n\Pi_{i=0}^{n-1}(\alpha-i).$
$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}(-1)^n\frac{\Pi_{i=0}^{n-1}(\alpha-i)}{n!}x^n.$

2.
$f(t) = \sqrt{t}$
$f'(t) = \frac{1}{2}t^{-1/2}, f''(t) = -\frac{1}{2^2}t^{-3/2}, f^{(n)}(t) = (-1)^{n-1}\frac{(2n-3)!!}{2^n}t^{1/2-n}. \ ((-1)!! \equiv 1)$
$f(t) = \sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(t-2)^n = \sum_{n=0}^{\infty}(-1)^{n-1}\frac{(2n-3)!!\sqrt{2}}{n!2^{2n}}(t-2)^n.$
If you set $x = 1-t$ and $\alpha = 1/2$ in problem 1, you will get the same series as problem 2.

3.
$\frac{d^2G(z,z')}{dz^2} = \delta(z-z'), G(0,z') = 0, G(H,z') = H.$
For $z < z', G_<(z,z') = az+b.$
For $z > z', G_>(z,z') = cz+d.$
$G(0,z') = G_<(0,z') = b = 0, G_<(z,z') = az.$
$G(H,z') = G_>(H,z') = cH+d = H, G_>(z,z') = c(z-H)+H.$
$G_>(z',z') = G_<(z',z'), az' = c(z'-H)+H.$
$\frac{dG(z,z')}{dz}|_{z'-\delta}^{z'+\delta} = 1, c-a = 1.$
$a = \frac{z'}{H}, c = 1+\frac{z'}{H}.$
Then $G_<(z,z') = \frac{zz'}{H}, G_>(z,z') = \left(1-\frac{z'}{H}\right)(z'-H)+H = (z-z')+\frac{zz'}{H}.$

4.
$T_p(z) = -\frac{1}{k}\left(\int_0^z G_>(z,z')Q(z')dz' + \int_z^H G_<(z,z')Q(z')dz'\right)$
$T_h(z) = Az+B$ which is the solution of $\frac{d^2T}{dz^2} = 0.$
$T(z) = T_p(z)+T_h(z)$ satisfies the boundary conditions $T(0) = 0, T(H) = T_0.$
Then we can determine A and B.
For simplicity, set $Q(z) = Q_0$ as textbook.
$T_p(z) = -\frac{Q}{k}\left(\int_0^z\left((z-z')+\frac{zz'}{H}\right)dz' + \int_z^H\frac{zz'}{H}dz'\right) = -\frac{Qz}{2k}(z+H)$
$T_h(z) = Az+B = \frac{T_0z}{H}+\frac{QHz}{k}$
$T(z) = \frac{T_0z}{H}+\frac{Qz}{2k}(H-z).$

5.
$G(\vec{r},t) = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{r}}G\left(\vec{k},t\right), \delta(r) = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{r}}.$
Insert these to the Schrodinger equation and get
$\frac{\hbar}{i}\frac{\partial G(\vec{k},t)}{\partial t}+\frac{\hbar^2}{2m}k^2G\left(\vec{k},\omega\right) = \delta(t),$

For $t > 0, G\left(\vec{k}, t\right) = Ae^{-i\frac{\hbar k^2}{2m}t}$.

For $t < 0, G\left(\vec{k}, t\right) = 0.$ (causal solution)

At $t \sim 0, \frac{\hbar}{i}\left(Ae^{-i\frac{\hbar k^2}{2m}0} - 0\right) = 1, A = \frac{i}{\hbar}$.

$G\left(\vec{r}, t\right) = \frac{i}{\hbar}\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} e^{-i\frac{\hbar k^2}{2m}t} = \frac{i}{\hbar}\frac{1}{(2\pi)^3}\int dk_x dk_y dk_z e^{i(k_x x + k_y y + k_z z) - i\frac{\hbar t}{2m}\left(k_x^2 + k_y^2 + k_z^2\right)}$

$= \frac{i}{\hbar}\frac{1}{(2\pi)^3}\left(\pi/\frac{i\hbar t}{2m}\right)^{3/2} \exp\left(-\frac{x^2+y^2+z^2}{4\frac{i\hbar t}{2m}}\right) = \frac{i}{\hbar}\frac{1}{(2\pi i\hbar t/m)^{3/2}} e^{\frac{imr^2}{2\hbar t}}$ where $r^2 = x^2 + y^2 + z^2$.

Note the Gaussian integral $\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \sqrt{\pi/a}e^{\frac{b^2}{4a}}$.