# Homework8

1.
$f(x) = \frac{1}{2}a_0 + \Sigma_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \Sigma_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$
$a_n = \frac{1}{L}\int_{-L}^{L} f(x') \cos\frac{n\pi x'}{L} dx', b_n = \frac{1}{L}\int_{-L}^{L} f(x') \sin\frac{n\pi x'}{L} dx'.$
Inserting $a_n$ and $b_n$ to $f(x)$, you will get
$f(x) = \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(x')\,dx'\right)$
$+\Sigma_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(x')\cos\frac{n\pi x'}{L}dx'\right)\cos\left(\frac{n\pi x}{L}\right)+\Sigma_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(x')\sin\frac{n\pi x'}{L}dx'\right)\sin\left(\frac{n\pi x}{L}\right)$
$= \int_{-L}^{L}\left[\frac{1}{L}\left(\frac{1}{2} + \Sigma_{n=1}^{\infty}\cos\frac{n\pi x'}{L}\cos\left(\frac{n\pi x}{L}\right) + \Sigma_{n=1}^{\infty}\sin\frac{n\pi x'}{L}\sin\left(\frac{n\pi x}{L}\right)\right)\right] f(x')\,dx'$
Therefore, $\frac{1}{L}\left(\frac{1}{2} + \Sigma_{n=1}^{\infty}\cos\frac{n\pi x'}{L}\cos\left(\frac{n\pi x}{L}\right) + \Sigma_{n=1}^{\infty}\sin\frac{n\pi x'}{L}\sin\left(\frac{n\pi x}{L}\right)\right)$
$= \frac{1}{L}\left(\frac{1}{2} + \Sigma_{n=1}^{\infty}\cos\frac{n\pi\left(x-x'\right)}{L}\right) = \delta(x-x')$

2.
$I = \int_0^{\infty}\frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{(x+ia)^2(x-ia)^2} = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$
$f(z) = \frac{1}{(z+ia)^2(z-ia)^2}$ with second-order poles $z = \pm ia$. We close the contour in the upper half plane and the reside will be $\frac{d}{dz}\left((z-ia)^2\frac{1}{(z+ia)^2(z-ia)^2}\right)|_{z=ia} = \left(\frac{-2}{(2ia)^3}\right) = \frac{1}{2\pi i}\frac{\pi}{2a^3}$. (We will get the same result by closing the contour in the lower half plane.) Therefore, $I = \frac{\pi}{4a^3}$.

3.
Let $z = e^{i\theta}.dz = ie^{i\theta}d\theta = izd\theta.d\theta = \frac{dz}{iz}.\cos\theta = \left(z+z^{-1}\right)/2.$
$I = \oint_{\|z\|=1}\frac{dz}{iz(1+\varepsilon(z+z^{-1})/2)} = \frac{2}{i}\oint_{\|z\|=1}\frac{dz}{\varepsilon z^2+2z+\varepsilon} = \frac{2}{i}\oint_{\|z\|=1}\frac{dz}{\varepsilon(z-z_+)(z-z_-)}$ with $z_\pm = \frac{-1\pm\sqrt{1-\varepsilon^2}}{\varepsilon}$. Only $z_+$ is inside the contour because $|z_+| < 1$.
Therefore, $I = 2\pi i\frac{2}{i\varepsilon}\frac{1}{z_+-z_-} = \frac{2\pi}{\sqrt{1-\varepsilon^2}}$.

4.
Insert $G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega G(\omega)\, e^{i\omega\left(t-t'\right)}$ and $\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega\left(t-t'\right)}$ into eq(3). We will get $(m(i\omega)+\beta)\, G(\omega) = 1$ so $G(\omega) = \frac{1}{im\omega+\beta}$. Then $G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\frac{1}{im\omega+\beta}e^{i\omega\left(t-t'\right)}$. The pole $\omega = \frac{i\beta}{m}$ lies in the upper plane. If $t-t' > 0$, we close the contour in the upper plane and $G(t,t') = 2\pi i\frac{1}{2\pi i m}e^{i\left(\frac{i\beta}{m}\right)\left(t-t'\right)} = \frac{1}{m}e^{-\frac{\beta}{m}\left(t-t'\right)}$. If $t-t' < 0$, we close the contour in the lower plane where no pole exists. So $G(t,t') = 0$ for $t-t' < 0$.

5.
For $t < t', G(t,t') = 0$.(Using causality, we can choose this boundary condition.)
For $t > t', m\frac{dG}{dt} + \beta G = 0, \int\frac{dG}{G} = \int_{t'}^{t}\left(-\frac{\beta}{m}\right)dt''$.Then $G = Ae^{\frac{-\beta}{m}\left(t-t'\right)}$.

For $t \sim t'$,Integrate over the region $(t' - \varepsilon, t' + \varepsilon)$, $\int_{t'-\varepsilon}^{t'+\varepsilon} m \frac{dG}{dt} dt + \int_{t'-\varepsilon}^{t'+\varepsilon} \beta G = \int_{t'-\varepsilon}^{t'+\varepsilon} \delta(t - t')$.

Then $m(G(t = t' + \varepsilon) - G(t = t' - \varepsilon)) = 1, A = \frac{1}{m}$.

Therefore, $G(t, t') = 0$ for $t < t'$ and $G(t, t') = \frac{1}{m} e^{\frac{-\beta}{m}(t - t')}$ for $t > t'$.