HW7

1.
$(u_n \cdot u_n) = A^2 \int_{-L}^{L} \cos(n\pi x/L)^2\, dx = A^2 \int_{-L}^{L} \frac{1+\cos(2n\pi x/L)}{2} dx = A^2 L = 1.$
So $A = 1/\sqrt{L}$.
$(u_n \cdot u_m) = \frac{1}{L}\int_{-L}^{L} \cos(n\pi x/L)\cos(m\pi x/L)\, dx$
$= \frac{1}{L}\int_{-L}^{L} \frac{1}{2}\left(\cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(n-m)\pi x}{L}\right)\right) dx = 0$ for $n \neq m$.
Therefore $(u_n \cdot u_m) = \delta_{nm}$.

2.
$f(x) = -x^2 + 1$ is an even function of x so only Fourier cosine terms survive.
$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$ with $a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\, dx = -\frac{4(-1)^n}{n^2\pi^2}$. (integrate by part twice)

3.$\psi\left(\vec{k}\right) = \frac{1}{(2\pi)^3}\int \lambda e^{-r/a} e^{-i\vec{k}\cdot\vec{r}} d^3r = \frac{\lambda}{(2\pi)^3}\int dr d\theta d\phi r^2 \sin\theta e^{-r/a} e^{-ikr\cos\theta}$
$= \frac{\lambda 2\pi}{(2\pi)^3}\frac{1}{ik}\int dr r e^{-r/a}\left(e^{ikr} - e^{-ikr}\right)$
$= \frac{\lambda 2\pi}{(2\pi)^3}\frac{1}{ik}\left(\frac{4ia^3k}{(1+a^2k^2)^2}\right) = \frac{\lambda a^3}{\pi^2(1+a^2k^2)^2}$. (integrate by part)

4.
If $k > 0$, then close the contour in the upper half plane. (the pole is ia)
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+a^2} e^{ikx} dx = \frac{1}{2\pi} 2\pi i \left(\frac{e^{ik(ia)}}{2ia}\right) = \frac{1}{2a} e^{-ka}$
If $k < 0$, then close the contour in the lower half plane. (the pole is $-ia$)
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+a^2} e^{ikx} dx = \frac{1}{2\pi}(-2\pi i)\left(\frac{e^{ik(-ia)}}{-2ia}\right) = \frac{1}{2a} e^{ka}$.
Therefore$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+a^2} e^{ikx} dx = \frac{1}{2a} e^{-|k|a}$.

5.
(1) $e^z = e^{x+iy} = e^x(\cos y + i\sin y) = (e^x \cos y) + i(e^x \sin y) = g + ih$
It satisfies Cauchy-Riemann condition at all points in the complex plane.
$\partial_x g = e^x \cos y = \partial_y h, \partial_y g = -e^x \sin y = -\partial_x h$. So e^z is analytic at $z = 0$.
(2) Let $f(z) = \sqrt{z}.f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$.
$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\sqrt{\Delta z}}$.
If we choose $\Delta z = \Delta x$, $f'(0) = \lim_{\Delta x \to 0} \frac{1}{\sqrt{\Delta x}} = \infty$. (approach the origin from real axis)
If we choose $\Delta z = i\Delta y$, $f'(0) = \lim_{\Delta y \to 0} \frac{1}{\sqrt{i\Delta y}} = \frac{1}{\sqrt{i}}\infty$. (approach the origin from the imaginary axis)
The two limits are different. Therefore $\sqrt{z}$ is not analytical at $z = 0$.

6.
$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, H(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t) e^{i\omega t} dt.$
$\int_{-\infty}^{\infty} F(\omega) H^*(\omega) e^{-i\omega t} d\omega = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 f(t_1) h^*(t_2) \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_1+t_2)}$
$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 f(t_1) h^*(t_2) 2\pi\delta(t - t_1 + t_2)$
$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau f(t_1) h^*(t+\tau)$

7.
$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(-\frac{1}{4}\right)\frac{e^{i2x}-2+e^{-i2x}}{x^2}dx = -\frac{1}{8}\left(\int_{-\infty}^{\infty}\frac{e^{i2x}-1}{x^2}dx + \int_{-\infty}^{\infty}\frac{e^{-i2x}-1}{x^2}dx\right)$
$= -\frac{1}{8}2\pi i\left(\frac{1}{2}\right) res\left[\frac{e^{i2x}-1}{x^2}\right] - \frac{1}{8}2\pi i\left(-\frac{1}{2}\right) res\left[\frac{e^{-i2x}-1}{x^2}\right]$
$= -\frac{1}{8}\pi i\,(2i) - \frac{1}{8}\pi i\,(2i) = \frac{\pi}{2}$

Note when calculating $\int_{-\infty}^{\infty}\frac{e^{i2x}-1}{x^2}dx$ you will close the contour in the upper half plane $\left(\lim_{x\to i\infty} e^{i2x} = 0\right)$. When calculating $\int_{-\infty}^{\infty}\frac{e^{-i2x}-1}{x^2}dx$ you will close the contour in the lower half plane $\left(\lim_{x\to -i\infty} e^{-i2x} = 0\right)$. The $\left(\pm\frac{1}{2}\right)$ factor is because you detour around the pole at $z = 0$ with counterclockwise/clockwise half circle .

8.

$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(k)\,e^{ikx}dk, \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}dk.$

Inserting these to the diffusion equation, you will get $\left(Dk^2 + K^2\right)\phi(k) = Q$. So $\phi(k) = \frac{Q}{(Dk^2+K^2)}$.

$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{Q}{(Dk^2+K^2)}\right)e^{ikx}dk = \frac{Q}{D}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{k^2+\left(K/\sqrt{D}\right)^2}e^{ikx}dk = \frac{Q}{D}\frac{1}{2K/\sqrt{D}}e^{-|x|K/\sqrt{D}}.$
(From Prob.4)