HW6

1.
$\hat{\phi} = \left(-\frac{y}{r}, \frac{x}{r}, 0\right), \vec{v} = \left(-\frac{y}{r}v\left(r\right), \frac{x}{r}v\left(r\right), 0\right).$
$\nabla \times \vec{v} = \left(\partial_x v_y - \partial_y v_x\right)\hat{z} = \left(\frac{1}{r}v\left(r\right) + x\partial_x\left(\frac{v(r)}{r}\right) + \frac{1}{r}v\left(r\right) + y\partial_y\left(\frac{v(r)}{r}\right)\right)\hat{z}$
$= \left(\frac{2v(r)}{r} + \partial_r\left(\frac{v(r)}{r}\right)\left(x\partial_x r + y\partial_y r\right)\right)\hat{z} = \left(\frac{v(r)}{r} + \partial_r v\left(r\right)\right)\hat{z}$

2.
(a) $\frac{v}{r} + v' = c$ implies $v\left(r\right) \sim r = \Omega r$ where Ω is a constant.
(b) $\nabla \times \vec{v} = \left(-\partial_y v_x\right)\hat{z} = -\partial_y\left(v_0 \exp\left(-y^2/L^2\right)\right)\hat{z} = \frac{2yv_0}{L^2}\exp\left(-y^2/L^2\right)\hat{z}$

3.
(i) $\int r\ \hat{\phi} \cdot \left(rd\phi\hat{\phi}\right) = 2\pi R^2$
(ii) $\nabla \times \vec{v} = \frac{1}{r}\partial_r\left(rv_\phi\right)\hat{z} = \frac{1}{r}\partial_r\left(r^2\right)\hat{z} = 2\hat{z}, \int \nabla \times \vec{v} \cdot d\vec{s} = 2\hat{z} \cdot \hat{z}\pi R^2 = 2\pi R^2$
(iii) $\int \nabla \times \vec{v} \cdot d\vec{s} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} 2\hat{z} \cdot \hat{r}R^2 \sin\theta d\theta d\phi = 2\pi R^2$

4.

The current density is $\vec{j} = I\delta\left(x\right)\delta\left(y\right)\hat{z}$ such that $\int \vec{j}.d\vec{a} = I$.
$\int \nabla \times \vec{B} \cdot d\vec{s} = \int \vec{B} \cdot d\vec{r} = \int B\hat{\phi} \cdot \hat{\phi}rd\phi = 2\pi rB = \mu_0 \int \vec{j} \cdot d\vec{s} = \mu_0 I.$
So $\vec{B} = \frac{\mu_o I}{2\pi r}\hat{\phi}$.
For $r \neq 0, \nabla \times \vec{B} = \hat{r}\left(-\partial_z B_\phi\right) + \hat{z}\frac{1}{r}\left(\partial_r\left(rB_\phi\right)\right) = 0.$
For $r = 0, \nabla \times \vec{B} \sim \delta\left(x\right)\delta\left(y\right) = A\delta\left(x\right)\delta\left(y\right).$ Integrate both sides to get A.
$A = \int \nabla \times \vec{B} \cdot d\vec{s} = \int \frac{\mu_o I}{2\pi r}\hat{\phi} \cdot \hat{\phi}rd\phi = \mu_o I.$
Therefore $\nabla \times \vec{B} = \mu_o I\delta\left(x\right)\delta\left(y\right)\hat{z} = \mu_0\vec{j}.$

5.
$f\left(x\right) = \frac{1}{1-x} = \sum_{n=0}^{\infty}\frac{f^{(n)}\left(\frac{1}{2}\right)}{n!}\left(x - \frac{1}{2}\right)^n$
$= 2 + 4\left(x - \frac{1}{2}\right) + 8\left(x - \frac{1}{2}\right)^2 + 16\left(x - \frac{1}{2}\right)^3 + ... = \sum_{n=0}^{\infty}2^{n+1}\left(x - \frac{1}{2}\right)^n.$
The radius of convergence is $\lim_{n-\infty}\left|\frac{c_n}{c_{n+1}}\right| = \frac{1}{2}$.

6.
$I = \int_0^\infty e^{-\lambda x^2}dx,$
$I^2 = \frac{1}{4}\int_{-\infty}^{\infty} e^{-\lambda x^2}dx\int_{-\infty}^{\infty} e^{-y^2}dy = \frac{1}{4}\int e^{-\lambda\left(x^2+y^2\right)}dxdy$
$= \frac{1}{4}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-\lambda r^2}drrd\theta = \frac{\pi}{4\lambda}.$So $I = \sqrt{\pi/\lambda}/2$
For $n = 2m$
$\int_0^\infty x^{2m}e^{-\lambda x^2}dx = \left(-1\right)^m \partial_\lambda^m \int_0^\infty e^{-\lambda x^2}dx = \frac{\sqrt{\pi}}{2}\frac{(2m-1)!!}{2^m}\lambda^{-\frac{2m+1}{2}}.$
Note $\left(2m - 1\right)!! = 1 \cdot 3 \cdot 5... \cdot \left(2m - 1\right); \left(-1\right)!! = 1.$
For $n = 2m + 1$
$\int_0^\infty x^{2m+1}e^{-\lambda x^2}dx = \frac{1}{2\lambda^{m+1}}\int_0^\infty y^m e^{-y}dy = \frac{1}{2\lambda^{m+1}}m!$ Note $y = \lambda x^2$.
$\int_0^\infty e^{-\lambda x^2 - \alpha x^4}dx = \int_0^\infty e^{-\lambda x^2}\sum_{n=0}^{\infty}\frac{\left(-\alpha x^4\right)^n}{n!}dx$

$=?\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\int_0^{\infty}e^{-\lambda x^2}x^{4n}dx=\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left(\frac{\sqrt{\pi}}{2}\frac{(4n-1)!!}{2^{2n}}\lambda^{-\frac{4n+1}{2}}\right)$
$=\sum_{n=0}^{\infty}\frac{(-\alpha)^n\sqrt{\pi}(4n-1)!!}{n!2^{2n+1}}\lambda^{-\frac{4n+1}{2}}$

The radius of convergence is $\lim_{n-\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n-\infty}\frac{4\lambda^2(n+1)}{a(4n+3)(4n+1)} = 0$. This indicates that summation of n and integration of x do not commutes for this sequence.

7.
$f(x,\varepsilon)=\frac{\varepsilon}{x^2+\varepsilon^2}$.
$f(x\neq 0,\varepsilon)\to\frac{\varepsilon}{x^2}\to 0, f(x=0,\varepsilon)\to\frac{1}{\varepsilon}\to\infty$
$\int_{-\infty}^{\infty}f(x,\varepsilon)\,dx=\pi$.
Therefore, $f(x,\varepsilon)=\pi\delta(x)$

8.
$\int_1^{\infty}\delta(\sin x)\,e^{-x}dx=\int_1^{\infty}e^{-x}\sum_i\frac{1}{\left|\frac{d\sin}{dx}(x_i)\right|}\delta(x-x_i)\,dx$
$=\int_1^{\infty}\sum_n\frac{1}{|\cos(n\pi)|}\delta(x-n\pi)\,e^{-n\pi}dx=\sum_{n=1}^{\infty}e^{-n\pi}=\frac{1}{e^{\pi}-1}$.
Note $x_i=n\pi$ such that $\sin(x_i)=0$.