Homework 3

1.

$$\vec{A}=\begin{pmatrix}1\\3\\2\end{pmatrix},\vec{B}=\begin{pmatrix}2\\4\\1\end{pmatrix}$$

$$\vec{C}=\vec{A}\vec{B}^T=\begin{pmatrix}1\\3\\2\end{pmatrix}\begin{pmatrix}2&4&1\end{pmatrix}=\begin{pmatrix}2&4&1\\6&12&3\\4&8&2\end{pmatrix},$$

$$\vec{C'}=\vec{B}\vec{A}^T=\begin{pmatrix}2\\4\\1\end{pmatrix}\begin{pmatrix}1&3&2\end{pmatrix}=\begin{pmatrix}2&6&4\\4&12&8\\1&3&2\end{pmatrix},$$

$$\vec{C}=\vec{A}\vec{B}^T=\left(\vec{B}\vec{A}^T\right)^T=\vec{C}'^T.$$

2.

Let $\vec{n}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$. The projection operator can be constructed by $P=\vec{n}\vec{n}^T=\frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1&1&1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$.

Method1. P will rescale the eigenvectors by eigenvalues $Pv_i=\lambda_i v_i$. P will not change $\vec{n}$. Therefore $\vec{n}$ is the eigenvector with eigenvalue 1. Let $\vec{t_1}$ and $\vec{t_2}$ be two vectors orthogonal to $\vec{n}$ $\left(\text{say, }\vec{t_1}=(1,-1,0)^T/\sqrt{2},\vec{t_2}=(1,1,-2)^T/\sqrt{6}\right)$. They have no projection on $\vec{n}$. When acted by P, you will get zero. Therefore the other two eigenvectors are $\vec{t_1}$ and $\vec{t_2}$ with eigenvector 0. (degeneracy)

Method2. Straightforward calculation of eigenvectors and eigenvalues.

$$\det\left(\frac{1}{3}\begin{pmatrix}1-3\lambda&1&1\\1&1-3\lambda&1\\1&1&1-3\lambda\end{pmatrix}\right)=0,\lambda^2(\lambda-1)=0.$$

$\lambda_1=1,\vec{v}_1=\vec{n}=(1,1,1)/\sqrt{3}$.

$\lambda_2=\lambda_3=0$ (degeneracy), we can choose two orthogonal vectors in this eigenspace, say $\vec{t_1}$ and $\vec{t_2}$.

3.

Let $\{\vec{e}_i\}$ be an orthonormal basis such that $\vec{e}_i\cdot\vec{e}_j=\delta_{ij}$. An arbitrary vector $\vec{A}$ can be expanded in the basis as $\vec{A}=\Sigma_i A_i\vec{e}_i$. By multiplying $\vec{e}_j^T$ both sides, we get $\vec{e}_j^T\cdot\vec{A}=\Sigma_i A_i\vec{e}_j^T\cdot\vec{e}_i=\Sigma_i A_i\delta_{ij}=A_j$. By substitutng it back to the expansion we have $\vec{A}=\Sigma_i\vec{e}_i^T\cdot\vec{A}\vec{e}_i=\left(\Sigma_i\vec{e}_i\vec{e}_i^T\right)\cdot\vec{A}$ for arbitrary $\vec{A}$. We get $\Sigma_i\vec{e}_i\vec{e}_i^T=I$.

4.

A matrix A satisfys the eigen-equation $A\vec{v}_i=\lambda_i\vec{v}_i$. We can construct $R=\begin{pmatrix}\vec{v}_1^T\\\vec{v}_2^T\\\vec{v}_3^T\end{pmatrix}$. Then $RAR^T=\begin{pmatrix}\vec{v}_1^T\\\vec{v}_2^T\\\vec{v}_3^T\end{pmatrix}\begin{pmatrix}A\vec{v}_1&A\vec{v}_2&A\vec{v}_3\end{pmatrix}=\begin{pmatrix}\vec{v}_1^T\\\vec{v}_2^T\\\vec{v}_3^T\end{pmatrix}\begin{pmatrix}\lambda_1\vec{v}_1&\lambda_2\vec{v}_2&\lambda_3\vec{v}_3\end{pmatrix}=$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = D.$$

Inversing the relation, we get $A = R^T DR$.(Note $RR^T = I$). Given eigenvalues and eigenvectors, we can construct $R = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$.

Then we have $A = R^T DR = \frac{1}{6} \begin{pmatrix} 11 & -4 & -1 \\ -4 & 14 & -4 \\ -1 & -4 & 11 \end{pmatrix}$.

Or from the text book, $A = \Sigma_i \lambda_i v_i v_i^T = 1 \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} + 2 \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} + 3 \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 11 & -4 & -1 \\ -4 & 14 & -4 \\ -1 & -4 & 11 \end{pmatrix}$.

5.

$\vec{r} = \begin{pmatrix} r\sin\theta\cos\phi & r\sin\theta\sin\phi & r\cos\theta \end{pmatrix}$

$\hat{r} = \frac{\partial}{\partial r}\vec{r} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$

$\hat{\theta} = \frac{\partial}{r\partial\theta}\vec{r} = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{pmatrix}$

$\hat{\phi} = \frac{\partial}{r\sin\theta\partial\phi}\vec{r} = \begin{pmatrix} -\sin\phi & \cos\phi & 0 \end{pmatrix}$

6.

Define $R = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$. $RR^T = 1$. We have $\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = R \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$. Then $\begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix} = R^{-1} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = R^T \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$.