Homework10

1.

Evaluate the integral $I(\vec{r})$ in the Cartesian coordinate so that $I(\vec{r})$ can be the product of three independent inegrals:

$I(\vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} e^{i\frac{\hbar k^2}{2m}t} = \frac{1}{(2\pi)^3}\int dk_x dk_y dk_z e^{i(k_x x + k_y y + k_z z) + i\frac{\hbar t}{2m}\left(k_x^2+k_y^2+k_z^2\right)}$

$= \frac{1}{(2\pi)^3}\left(\pi/\frac{-i\hbar t}{2m}\right)^{3/2}\exp\left(-\frac{x^2+y^2+z^2}{-4\frac{i\hbar t}{2m}}\right) = \frac{1}{(-2\pi i\hbar t/m)^{3/2}} e^{-\frac{imr^2}{2\hbar t}}$

where $r^2 = x^2+y^2+z^2$.

Note the Gaussian integral $\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \sqrt{\pi/a}e^{\frac{b^2}{4a}}$.

2. Cosider two boundary conditions :

(a) $t < t'$, $G(t,t') = 0$, (b) $t > t'$, $G(t,t') = 0$

(a)

For $t < t'$, $G(t,t') = 0$

For $t > t'$, $\frac{\partial G(t,t')}{\partial t} + \beta G(t,t') = 0, G(t,t') = Ae^{-\beta t}$

For $t \sim t'$, $\int_{t'-\varepsilon}^{t'+\varepsilon} dt \left(\frac{\partial G(t,t')}{\partial t} + \beta G(t,t')\right) = \int_{t'-\varepsilon}^{t'+\varepsilon} dt \delta(t-t')$,

$G(t'+\varepsilon,t') - G(t'-\varepsilon,t') = 1, Ae^{-\beta t'} - 0 = 1, A = e^{\beta t'}$

Therefore, $G(t,t') = e^{\beta(t'-t)}\theta(t'-t)$.

(b)

For $t > t'$, $G(t,t') = 0$

For $t < t'$, $\frac{\partial G(t,t')}{\partial t} + \beta G(t,t') = 0, G(t,t') = Ae^{-\beta t}$

For $t \sim t'$, $\int_{t'-\varepsilon}^{t'+\varepsilon} dt \left(\frac{\partial G(t,t')}{\partial t} + \beta G(t,t')\right) = \int_{t'-\varepsilon}^{t'+\varepsilon} dt \delta(t-t')$,

$G(t'+\varepsilon,t') - G(t'-\varepsilon,t') = 1, 0 - Ae^{-\beta t'} = 1, A = -e^{\beta t'}$

Therefore, $G(t,t') = -e^{\beta(t'-t)}\theta(t-t')$.

Then $g(t,t') = e^{\beta(t'-t)}\theta(t'-t) - \left(-e^{\beta(t'-t)}\theta(t-t')\right) = e^{\beta(t'-t)}$

It satisfies $\frac{\partial g(t,t')}{\partial t} + \beta g(t,t') = 0$.

Note: $\theta(x-y) = 1$ if $x > y$, $\theta(x-y) = 0$ if $x < y$.

3.

Using separation of variables: let $\psi(x,y) = X(x)Y(y)$ and insert it into the Helmholtz equation. $\frac{d^2X(x)}{dx^2}Y(y) + X(x)\frac{d^2Y(y)}{dy^2} + k^2X(x)Y(y) = 0$. Divide it by $\psi(x,y)$: $\frac{\frac{d^2X(x)}{dx^2}}{X(x)} = -\frac{\frac{d^2Y(y)}{dy^2}}{Y(y)} - k^2 \equiv -k_x^2$, $\frac{\frac{d^2Y(y)}{dy^2}}{Y(y)} = -(k^2-k_x^2) \equiv -k_y^2$. The solution for these two equations will be $X(x) = A\cos(k_x x) + B\sin(k_x x)$, $Y(y) = C\cos(k_y y) + D\sin(k_y y)$. The coefficients are determined by the boundary conditions $X(0) = X(L_x) = 0, Y(0) = Y(L_y) = 0$. You will get $A = C = 0, k_x L_x = n_x\pi, k_y L_y = n_y \pi$ where n_x and n_y are integers. Therefore the normal modes will be $\psi(x,y) = E\sin(k_x x)\sin(k_y y)$ with $k_x = \frac{n_x\pi}{L_x}, k_y = \frac{n_y\pi}{L_y}$ and $k^2 = k_x^2 + k_y^2$. Here E can be determined by normalization condition

$\int_0^{L_x} dx \int_0^{L_y} dy \psi(x,y)^2 = 1$ and you will get $E = \frac{2}{\sqrt{L_x L_y}}$. Therefore the normal modes are $\psi(x,y) = \frac{2}{\sqrt{L_x L_y}} \sin(k_x x)\sin(k_y y)$.

4.

Now prove $\int_0^R \int_0^{2\pi} u_{n1m1}^*(r,\phi)\, u_{n2m2}(r,\phi)\, d\phi r dr = C_{n1n2}\delta_{n_1 n2}\delta_{m1m2}$ where $u_{nm}(r,\phi) = J_m\left(k_n^{(m)} r\right) e^{im\phi}$ and C_{n1n2} is a constant depends on n_1 and m_1.

For $e^{im\phi}$, $\int_0^{2\pi} e^{-i(m_1-m_2)\phi} d\phi = 2\pi\delta_{m1m2}$.

For $J_m\left(k_n^{(m)} r\right)$, start from the Bessel's equations with $k_{n1}^{(m)}$ and $k_{n2}^{(m)}$

(a) $r\frac{d^2}{dr^2} J_m\left(k_{n1}^{(m)} r\right) + \frac{d}{dr} J_m\left(k_{n1}^{(m)} r\right) + \left(k_{n1}^{(m)2} r - \frac{m^2}{r}\right) J_m\left(k_{n1}^{(m)} r\right) = 0,$

(b) $r\frac{d^2}{dr^2} J_m\left(k_{n2}^{(m)} r\right) + \frac{d}{dr} J_m\left(k_{n2}^{(m)} r\right) + \left(k_{n2}^{(m)2} r - \frac{m^2}{r}\right) J_m\left(k_{n2}^{(m)} r\right) = 0.$

Multiply (a) by $J_m\left(k_{n2}^{(m)} r\right)$ and (b) by $J_m\left(k_{n1}^{(m)} r\right)$ and subract : $J_m\left(k_{n2}^{(m)} r\right)(a) - J_m\left(k_{n1}^{(m)} r\right)(b)$:

$J_m\left(k_{n2}^{(m)} r\right)\left(r\frac{d^2}{dr^2} J_m\left(k_{n1}^{(m)} r\right) + \frac{d}{dr} J_m\left(k_{n1}^{(m)} r\right)\right) - J_m\left(k_{n1}^{(m)} r\right)\left(r\frac{d^2}{dr^2} J_m\left(k_{n2}^{(m)} r\right) + \frac{d}{dr} J_m\left(k_{n2}^{(m)} r\right)\right)$
$= \left(k_{n2}^{(m)2} - k_{n1}^{(m)2}\right) r J_m\left(k_{n1}^{(m)} r\right) J_m\left(k_{n2}^{(m)} r\right)$

$J_m\left(k_{n2}^{(m)} r\right) \frac{d}{dr}\left(r\frac{d}{dr} J_m\left(k_{n1}^{(m)} r\right)\right) - J_m\left(k_{n1}^{(m)} r\right) \frac{d}{dr}\left(r\frac{d}{dr} J_m\left(k_{n2}^{(m)} r\right)\right)$
$= \left(k_{n2}^{(m)2} - k_{n1}^{(m)2}\right) r J_m\left(k_{n1}^{(m)} r\right) J_m\left(k_{n2}^{(m)} r\right).$

Integrating from $r = 0$ to $r = R$:

$\int_0^R J_m\left(k_{n2}^{(m)} r\right) \frac{d}{dr}\left(r\frac{d}{dr} J_m\left(k_{n1}^{(m)} r\right)\right) dr - \int_0^R J_m\left(k_{n1}^{(m)} r\right) \frac{d}{dr}\left(r\frac{d}{dr} J_m\left(k_{n2}^{(m)} r\right)\right) dr$
$= \left(k_{n2}^{(m)2} - k_{n1}^{(m)2}\right) \int_0^R r J_m\left(k_{n1}^{(m)} r\right) J_m\left(k_{n2}^{(m)} r\right) dr.$

Upon integrating by parts

RHS$=\left[J_m\left(k_{n2}^{(m)} r\right) r\frac{d}{dr} J_m\left(k_{n1}^{(m)} r\right)\right]_0^R - \left[J_m\left(k_{n1}^{(m)} r\right) \frac{d}{dr}\left(r\frac{d}{dr} J_m\left(k_{n2}^{(m)} r\right)\right)\right]_0^R =$
0 because r guarantees a zero at the lower limit $r = 0$ and $k_{n1}^{(m)}, k_{n2}^{(m)}$ are roots of J_m, that is, $J_m\left(k_{n1/n2}^{(m)} R\right) = 0$.

LHS$=\left(k_{n2}^{(m)2} - k_{n1}^{(m)2}\right) \int_0^R r J_m\left(k_{n1}^{(m)} r\right) J_m\left(k_{n2}^{(m)} r\right) dr = 0.$

If $m \neq n$, $\int_0^R r J_m\left(k_{n1}^{(m)} r\right) J_m\left(k_{n2}^{(m)} r\right) dr = 0$. This gives us orthogonality over the interval $[0, R]$.

Now we have proved $\int_0^{2\pi} e^{-i(m_1-m_2)\phi} d\phi = 2\pi\delta_{m1m2}$ and $\int_0^R r J_m\left(k_{n1}^{(m)} r\right) J_m\left(k_{n2}^{(m)} r\right) dr = C'\delta_{n1n2}$ where C' normalized constant. Therefore, $\int_0^R \int_0^{2\pi} u_{n1m1}^*(r,\phi)\, u_{n2m2}(r,\phi)\, d\phi r dr = C_{n1n2}\delta_{n_1 n2}\delta_{m1m2}$.

5.

Expand a function $f(r,\phi)$ in terms of $u_{nm}(r,\phi)$:

$f(r,\phi) = \sum_{nm} d_{nm} u_{nm}(r,\phi)$ where $u_{nm}(r,\phi) = J_m\left(k_n^{(m)} r\right) e^{im\phi}$.

Using the orthogonal relation in problem4. $\int_0^R \int_0^{2\pi} u^*_{n1m1}(r,\phi)\, u_{n2m2}(r,\phi)\, d\phi r dr = C_{n1n2}\delta_{n_1n2}\delta_{m1m2}$, we can get $d_{nm} = \frac{1}{C_{nm}} \int_0^R \int_0^{2\pi} u^*_{nm}(r,\phi)\, f(r,\phi)\, d\phi r dr$. Insert this into $f(r,\phi)$, we will get

$f(r,\phi) = \sum_{nm} \left(\frac{1}{C_{nm}} \int_0^R \int_0^{2\pi} u^*_{nm}(r',\phi')\, f(r',\phi')\, d\phi' r' dr' \right) u_{nm}(r,\phi)$

$= \int_0^R \int_0^{2\pi} d\phi' r' dr' \left[\sum_{nm} \frac{1}{C_{nm}} u^*_{nm}(r',\phi')\, u_{nm}(r,\phi) \right] f(r',\phi')$.

Therefore, $\sum_{nm} \frac{1}{C_{nm}} u^*_{nm}(r',\phi')\, u_{nm}(r,\phi) = \frac{1}{r}\delta(r-r')\,\delta(\phi-\phi')$.