Physics 374 Take Home Exam
Due October 27 (11:00 A.M.)

This is a take home exam and not a typical problem set. Accordingly you may not seek help from others. You are permitted to consult your class notes and reference books and you are permitted to use Mathematica or some other symbolic manipulator. If you do use Mathematica, you must include a printout of your computations. If you are unclear about the exam, please contact me either by e-mail or phone (301-405-6117 (office) or 301-654-7702 (home)). Do not call me at home after 10:00 p.m. Before calling me please check the course's web site. I will post any clarification or corrections based on other student's questions or comments. As you have a long time to do this exam, I fully expect all your answer's to be explained clearly. You will not receive credit unless you explain where your results come from based on the physical and mathematical principles we have developed this semester. You may however take as given any formulae derived in class this semester---you need not reinvent the wheel on every problem.

Each lettered section is of equal worth.

Some of these problems are very straightforward but some are less so.
Do not panic if you do not get everything. Note also that in some questions, I have asked you to show a certain answer is correct rather than to simply solve the problem. I have done this largely so that you can use the result in other parts of the problem even if you cannot derive it yourself.

Clearly, on a take home exam I cannot efficiently monitor whether students are cheating. Instead, I rely largely on your integrity. Ultimately, the scientific enterprise depends on trusting the integrity of scientists; when I read a paper I need to be morally certain that the authors have not cooked the data. Thus, on this exam I am treating you as I would any scientist and trusting your integrity. Do not violate this trust.

1) Consider a particle of mass, m, moving in a harmonic oscillator potential of the form $U(x) = \dfrac{m\omega_0^2}{2}x^2$. In addition the mass experiences an external force $F(t) = F_0 \cos(2\omega_0 t)$. The particle is released from rest at the origin. time t=0. The purpose of this problem is to develop an approximate solution to this problem which is valid in the regime where the displacement is sufficiently small so that harmonic oscillator potential is small and to compare this solution to the exact solution.

   a) Show from Newtonian physics that the equation of motion can be written in the following form $\ddot{x}(t) = -\omega_0^2 x(t) + \omega_0^2 L\cos(2\omega_0 t)$ where L is a characteristic length in the problem. Find L in terms of the original parameters in the problem.

   b) We wish to study the regime of small displacements accordingly it is useful to rewrite the differential equation as $\ddot{x}(t) = -\lambda\omega_0^2 x(t) + \omega_0^2 L\cos(2\omega_0 t)$ where λ is put in to keep track of smallness (it is set to one at the end of the problem). The solution can be written in the form: $x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \lambda^3 x_3(t) + \dots$. Show that the equations for the $x_n(t)$ are given by:

$$\ddot{x}_0(t) = \omega_0^2 L\cos(2\omega_0 t) \qquad \ddot{x}_1(t) = -\omega_0^2 x_0(t) \qquad \ddot{x}_2(t) = -\omega_0^2 x_1(t) \quad .... \quad \ddot{x}_n(t) = -\omega_0^2 x_{n-1}(t)$$

c) Solve for $x_0(t)$ subject to the boundary conditions $x_0(0) = 0$; $\dot{x}_0(0) = 0$. (The correct answer is given by $x_0(t) = \frac{L}{4}(1 - \cos(2\omega_0 t))$; if you are unable to get this you may use the correct result in the following section.)

d) What are the appropriate boundary conditions for $x_1(t)$ and $x_2(t)$? Why are they appropriate? Now solve for $x_1(t)$ and $x_2(t)$ subject to these boundary conditions and write an approximate solution for the problem valid up to order λ^2.

e) The exact solution for $x(t)$ (keeping the factor of λ as given in part b) can be shown to be given by $x(t) = \frac{L}{(4-\lambda)}\left(\cos(\lambda^{1/2}\omega_0 t) - \cos(2\omega_0 t)\right)$. This can easily be verified by direct substitution. (**Note you do not need to verify this to get credit on this question**). Show that the exact result in is consistent with the expansion in parts c) and d

2) Consider a particle of mass, m, moving in a potential given by the potential $U(x) = \frac{\alpha x^4}{24}$ where α is a parameter. Suppose a particle is released at rest from an initial displacement A.

a) Use dimensional analysis to find a combination of the parameters in this problem which gives the characteristic time associated with the problem, t_c. Explain on dimensional grounds why one expects that the period of oscillation to be $\tau = c\, t_c$ where c is a purely numerical constant.

b) Use energy conservation to show that the period oscillation is given by
$$\frac{(8\sqrt{3})\sqrt{\dfrac{m}{\alpha}}\displaystyle\int_0^1 dy\,\dfrac{1}{\sqrt{1-y^4}}}{A}.$$ Note the integral can be done numerically: Its value is approximately given by $\displaystyle\int_0^1 dy\,\frac{1}{\sqrt{1-y^4}} \approx 1.311$ (**You do not need to do this integral to get credit**).

3) Consider a particle of mass, m, moving in a potential given by the potential $U(x) = V_0 \sin^2(x/L)$ where V_0 and L are parameters. In this problem we consider the motion of a particle moving near the origin (namely $|x|<<L$). Given this you may find it useful to expand the potential

a) Find an expression for the frequency of oscillation for arbitrarily small amplitude oscillations (Hint: this is as easy at seems, the system effectively acts as a harmonic oscillator).

b) Find an expression for the first non-vanishing correction to the result in a) due to anharmonicity. (Hint: you do not need to redo the Fourier analysis that we did in class; it legitimate to steal those results.)

4) Consider a particle moving in some inertial frame (which we denote the "lab frame") whose position as a function of time is given by $x(t) = \frac{1}{a}\sqrt{1+(at)^2}$ where a is a parameter and we work in n units where $c=1$. The purpose of this problem is to describe the motion both in its original frame in a frame (the primed frame) moving to the right with a velocity of v.

a) The problem is awkward to deal with as we do not have the trajectories expressed in a covariant way (*i.e.* as part of a 4-vector). As an intermediate step we will want to express the path as a four-vector $\vec{x}^4$ parameterized by the proper time. To do this we will need to find τ as a function of t. Show that $\tau = \frac{1}{a}\sinh^{-1}(at)$. If you cannot get this result just assume the answer and go on.

b)   Show that the position 4-vector in the lab frame is $\vec{x}^4 = \begin{pmatrix} a^{-1}\sinh(a\tau) \\ a^{-1}\cosh(a\tau) \\ 0 \\ 0 \end{pmatrix}$

c) Find the four-velocity $\vec{u}^4$ and four acceleration $\vec{a}^4$ and verify explicitly that $\vec{u}^4 \cdot \vec{u}^4 = 1 \quad \vec{a}^4 \cdot \vec{u} = 0$.

d) Using the result of part b) find the position 4-vector in the primed frame.

e) Verify explicitly that $\vec{x}^4 \cdot \vec{x}^4 = \vec{x}'^4 \cdot \vec{x}'^4$, *i.e.* that the Lorentz dot product of $\vec{x}^4$ with itself is the same in both frames (as it should be since it is a Lorentz scalar.

5) A dispersion relation relates the frequency and wave number of a wave. Consider the following dispersion relation: $\omega = \sqrt{k^2 + \mu^2}$ where k is the magnitude of $\vec{k}$ (*i.e.* $k^2 \equiv \vec{k} \cdot \vec{k}$). (Cultural note: this kind of dispersion relation arises naturally in relativistic quantum field theory in which case the wave is associated with the propagation of particles of mass μ as given in units where the speed of light and planck's constant are both set to unity) . To avoid confusion let me point out explicitly that with our notation $\vec{k}$ is an ordinary Euclidean three vector. We will find it useful to consider the 4-vector associated with such a wave:

$$\vec{k}^4 = \begin{pmatrix} \omega \\ k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \sqrt{k^2+\mu^2} \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

a) Prove that μ is a Lorentz scalar (*i.e.* the same in all frames) by considering the Lorentz dot product of $\vec{k}^4$ with itself.

Suppose that in a given frame (which we will call the "lab" frame) the wave propagates in the positive x direction with a magnitude of k so that the four velocity is given by

$$\vec{k}^4 = \begin{pmatrix} \omega \\ k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \sqrt{k^2 + \mu^2} \\ k \\ 0 \\ 0 \end{pmatrix}$$

I would like to describe this in a frame moving to the right at a speed of v which will be denoted as the primed frame.

b) By boosting $\vec{k}^4$ to the new frame you can find the frequency in the new frame, ω' (*i.e.* you can find the Doppler shift) and the wave number in the new frame k'. Find ω' and k'. Demonstrate explicitly that in this new frame the dispersion relation is satisfied (*i.e.* show that $\omega' = \sqrt{k'^2 + \mu^2}$).

The group velocity of a wave is the speed at which a wave packet travels and is associated with rate at which physical quantities such as energy or momentum are carried by the wave. Recall from your wave's course that it is given by $v_g = \dfrac{d\omega}{dk}$ which for this dispersion relation gives $v_g = \dfrac{k}{\sqrt{k^2 + \mu^2}} = \dfrac{k}{\omega}$.

c) Using the results of part b) and the expression for the group velocity given above, show that the group velocity in the boosted frame is related to the group velocity in the initial frame by the usually velocity addition formula: $v_g' = \dfrac{v_g - v}{1 - v_g v}$ (where the minus sign is because we are boosting in the direction of propagation)

d) The Lorentz dot product of $\vec{k}^4$ with itself is positive. This means the $\vec{k}^4$ is a time- like vector, which in turn means that there exists a frame in which all the spatial components vanish ($k'_x = k'_y = k'_z = 0$ where the prime indicates this special frame). Show that in this frame $\omega' = \mu$. Hint: You can prove this without making any explicit Lorentz transformation. You will find it profitable to focus on the Lorentz dot product of $\vec{k}^4$ with itself and to consider how this looks in a frame where spatial components vanish.