Electrostatics —

basic laws

$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\nabla \Phi$

$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ using Gaussian units (cgs w/ Lo

if you insist on SI units

let $\rho \to \frac{\rho}{4\pi\epsilon_0}$

combine $\vec{\nabla} \cdot \vec{\nabla}\Phi = \nabla^2 \Phi = -4\pi\rho$ (Poisson equation)

Electrostatics in 2-Dimensional world

two views
- artificial world of 2 spatial direction
- 3-d world with all quantities independent of z. Eg. $\rho(\vec{x}) = \rho(x,y)$

by symmetry $E_z = 0$ in such a case

get a 2-d world

eg rods

why do this ?
- problems which are effectively 2-d do arise (effectively independent o
- much easier in certain technical ways than 3-D electrostatics
- same basic issues as 3-D

in 2-D problem no z dependence

$\Phi(x,y,z) \to \Phi(x,y)$

$\rho(x,y,z) \to \rho(x,y)$

so Poisson equation

$$\nabla^2 \Phi = -4\pi\rho$$

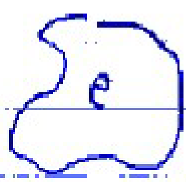
goes to 2-D poisson equation

$\nabla^2 \Phi = -4\pi\rho$ (all of Electrostatics)

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

not $\frac{\partial^2}{\partial z^2}$ is gone since Φ has no z dependence

problem to focus on



charge density is Localized to some region in space but zero beyond some range (or very small) natural to set origin in region near center of charge and to work in Polar coordinates

$x = r\cos\theta$ $\qquad\qquad$ $r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\qquad\qquad$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

1st step: write 2-D Laplacian in Polar coordinates

slightly tedious excercise —

work it out or look it up

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

so equation of Electrostatics

Poisson

$$\nabla^2 \Phi = -4\pi\rho$$

2-D Position

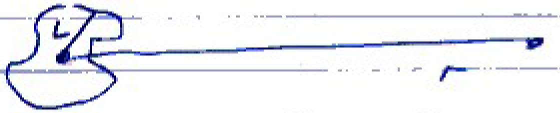
$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right]\Phi(\vec{x}) = -4\pi\rho(\vec{x})$$

Trick — expand as a Fourier series in polar coordinate ϕ since $\Phi(r,\phi) \equiv \Phi(r,\phi+2\pi)$
$\rho(r,\phi) \equiv \Phi(r,\phi+2\pi)$

$$\rho(\vec{x}) = \rho(r,\phi) = \sum_m \rho_m(r)\, e^{im\phi} \;;\; \rho_m(r) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \rho(\phi)\, e^{-im\phi}$$

$$\Phi(x) = \Phi(r,\phi) = \sum_m \Phi_m(r)\, e^{im\phi} \;;\; \Phi_m(r) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \Phi(\phi)\, e^{-im\phi}$$

- turns out to be basis of the multipole expansion
- useful in Φ converges with a few terms. We will see this is true far from charges



turns out to be a series in $\frac{r}{L}$

- of course Φ, ρ are real

this constrains

$$\Phi_m^*(r) = \Phi_{-m}(r)$$
$$\rho_m^*(r) = \rho_{-m}(r)$$

Proof: $\rho^*(r) = \rho(r)$

$$\rho^*(r) = \sum_m \rho_m^*(r) e^{-im\phi}$$

$$\rho(r) = \sum_n \rho_n(r) e^{in\phi}$$

let $n = -m$ (dummy variabl

$$= \sum_m \rho_{-m}(r) e^{-im\phi}$$

matching Fourier components

$$\rho_{-m}(r) = \rho_m^*(r)$$

analogous proof for Φ

Plug form into Laplace equation

$$\nabla^2 \Phi = -4\pi \rho$$

$$\left(\frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2}\partial_\phi^2\right)\sum_m \Phi_m(r)\, e^{im\phi} = -4\pi \sum_m \rho_m(r)\, e^{im\phi}$$

$$= \sum_m \left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right]\Phi_m(r)\, e^{im\phi} = -4\pi \sum_m \rho_m(r)\, e^{im\phi}$$

$$= \sum_m \left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2}\right]\Phi_m(r)\, e^{im\phi} = -4\pi \sum_m \rho_m(r)\, e^{im\phi}$$

Now multiply both sides by $e^{-in\phi}$ and then integrate with respect to ϕ (picks out one m)

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-in\phi} \sum_m \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r}\right)\Phi_m(r)\, e^{im\phi} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-in\phi} \sum_m \rho_m e^{im\phi}$$

use fact $\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\phi} = \delta_{mn}$

$$\sum_m \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r}\right)\Phi_m(r)\,\delta_{mn} = \sum_m \rho_m(r)\,\delta_{mn}$$

$$\boxed{\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{n^2}{r}\right]\Phi_n(r) = -4\pi \rho_n(r)}$$

P.D.E splits into uncoupled O.D.E's (Laplacian is separable in polar coordinates before solving look at asymptotics beyond range of charge $r > L$

$$\nabla^2 \Phi = 0$$

or for each m

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2}\right] \Phi_m(r) = 0$$

for each m there are two solutions

$m = 0$ $\qquad a_0 \quad$ const

$\qquad b_0 \operatorname{Log}(r)$

verify each is a solution

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\right] a_0 = 0 \quad \checkmark$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\right] b_0 \operatorname{Log}(r) =$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{b}{r} = \frac{1}{r}\frac{\partial}{\partial r} b = 0 \quad \checkmark$$

$m > 0$ $\qquad a_m r^{-m} + b_0 r^m$

verify

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\right) a_m r^{-m} =$$

$$\frac{1}{r}\frac{\partial}{\partial r} r\left(-m\, a_m r^{-m-1}\right) =$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(-m a_m r^{-m}\right) =$$

$$+ m^2 a_m r^{-m}$$

so $\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - m^2\right] a_m r^{-m} = 0$ ✓

similarly with $b_m r^{+m}$

for $m<0$ $a_{-m} r^{-|m|} + b_m r^{-|m|}$

General Form:

$$\Phi_m(r,\phi) = b_0 \log(r/r_0)$$

$$+ \sum_{m>0} a_m e^{im\phi} r^{-m} + c.c.$$

$$+ \sum_{m>0} b_m e^{im\phi} r^{+m} + c.c.$$

Claim on physical Grounds for $r>L$
all $b_m = 0$ why?
correspond to E fields growing at ∞
but E fields die away

$$\Phi_n(r,\phi) = b_0 \log(r/r_0) + \sum_{n>0} 2\,\mathrm{Re}\left[a_n e^{in\phi}\right] r^{-n}$$

$$= b_0 \log(r/r_0) + \sum_{n>0} \left(c_n \cos(n\phi) + s_n \sin(n\phi)\right) r^{-n}$$

with $c_n = 2\,\mathrm{Re}\left[a_n\right]$

$s_n = -2\,\mathrm{Im}\left[a_n\right]$

important point — each different ϕ dependence has different r dependence

— part which independent of ϕ goes like $\log(r)$ ~~goes~~ (monopole piece)

— part which goes as $\cos(\phi)$ or $\sin\phi$ goes like $\frac{1}{r}$ (dipole)

etc

why is this useful —

dimensional analysis

q_n (or c_n, s_n) has units of $\frac{\text{charge}}{\text{distance}}$
but only distance scale in problem
is L (r is not a scale)

so $\frac{q_{n+1}}{q_n} \sim L$

Look at series as $r \to \infty$
terms which have large n go to zero quickly

typically the $n+1$ first term will be $\frac{L}{r}$ smaller than the n^{th} term

as $r \to \infty$ will be dominated by lowest non vanishing n

Series is useful for $r \gg L$

A few useful points

— at large r a few coefficients can give all significant info about system

— these coefficients are called the multipole moments

— you don't need full details of $\rho(x,y)$ to describe system at large r only a few coef.

— get coef. theoretically from knowledge of $\rho(x,y)$ or experimentally by measuring a few numbers

Next: How do we extract multipole moments from $\rho(x,y)$

recall

$$\left(\frac{1}{r}\partial_r r \partial_r - \frac{m^2}{r^2}\right)\Phi_m(r) = \rho_m(r)$$

Claim – Solution is easy
(once you know it)

inspired guess

$$\Phi_m = \frac{2\pi}{m\, r^m}\int_0^r dr'(r')^{m+1}\rho_m(r') + \frac{2\pi}{m} r^m \int_r^\infty dr' \frac{\rho_m(r')}{r'^{(m-1)}}$$

plug in to check:

$$\frac{\partial}{\partial r}\Phi_m = \frac{-2\pi}{r^{m+1}}\int_0^r dr'(r')^{(m+1)}\rho_m(r') + \frac{2\pi}{m}\cancel{r\rho_m(r)}$$

$$+ 2\pi\, r^{(m-1)}\int_r^\infty dr' \frac{\rho_m(r')}{r'^{(m-1)}} - \frac{2\pi}{m}\cancel{r\rho_m(r)}$$

$$r\frac{\partial}{\partial r}\Phi_m = \frac{-2\pi}{r^m}\int_0^r dr'(r')^{m+1}\rho_m(r') + 2\pi r^m\int_r^\infty dr' \frac{\rho_m(r')}{r'^{(m-1)}}$$

$$\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\Phi_m = +\frac{2\pi m}{r^{m+1}}\int_0^r dr'(r')^{m+1}\rho_m(r') - 2\pi r\rho(r)$$

$$+ 2\pi m\, r^{(m-1)}\int_r^\infty dr' \frac{\rho(r')}{r'^{(m-1)}} - 2\pi r\rho(r)$$

so

$$\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\Phi_m = \frac{m^2}{r^2}\Phi_m - 4\pi\rho(r)$$

or

$$\boxed{\left[\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r} - \frac{m^2}{r^2}\right]\Phi_m = -4\pi\rho(r)}$$

Q.E.D.

Actually this is <u>not</u> the general solution. It works only for $n \neq 0$. Note $\frac{1}{n}$ in solution

what about $n=0$

$$\Phi_0 = -4\pi \log(r/r_0) \int_0^r dr' r' \rho_0(r') - 4\pi \int_r^\infty dr' r' \log(r'/r_0) \rho_0(r')$$

Plug and chug to check as in $n \neq 0$ case

– what about r_0?

- needed to argument of log dimensionless
- fixes the zero point of Φ
  (convention of $\Phi = 0$ as $r \to \infty$ fails for case where $\int_0^\infty dr' r' \rho_0(r') \neq 0$ as $E \sim \frac{1}{r}$ $\Phi \sim \log(r)$)
- changing r_0 just adds an overall constant to Φ_0 and hence does <u>not</u> alter $\vec{E}$ field

Summary — General solution

$$\vec{E} = -\nabla\phi \qquad\qquad \vec{\nabla}\Phi = \hat{r}\,\frac{\partial\Phi}{\partial r} + \hat{\phi}\,\frac{1}{r}\frac{\partial\Phi}{\partial\phi}$$

$$\Phi(r) = \sum_m \Phi_m(r)\, e^{+im\phi}$$

$$\Phi_m = \begin{cases} \frac{2\pi}{m r^m}\int_0^r dr'\,(r')^{(m+1)}\rho_m(r') + \frac{2\pi}{m} r^m \int_r^\infty dr'\,\frac{\rho(r')}{r'^{(m-1)}} & \text{for } m \neq 0 \\ -4\pi \log\left(\frac{r}{r_0}\right)\int_0^r dr'\,r'\,\rho_0(r') - 4\pi\int_r^\infty dr'\,r'\,\log(r'/r_0)\,\rho_0(r') & \text{for } m = 0 \end{cases}$$

$$\rho_m(r) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-im\phi}\,\rho(\vec{x})$$

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Return to large r region

$r \gg L$

$$\Phi(r) = b_0 \operatorname{Log}(r/r_0) + \sum_{m>0} r^{-m}\left[q_m e^{im\phi} + q_m^* e^{-im\phi}\right]$$

match general solution

integrate over whole source as we are outside

Aipot cets

$$\boxed{b_0 = -4\pi\int_0^\infty dr'\,r'\,\rho_0(r')}$$

$$\boxed{q_m = \frac{2\pi}{m}\int_0^\infty dr'\,(r')^{m+1}\rho_m(r')}$$

area alone

$$b_0 = -4\pi\int_0^\infty dr'\,r'\,\frac{1}{2\pi}\int_0^{2\pi} d\phi\,\rho(\vec{r}',\phi) = \boxed{-2\int d^2x'\,\rho(\vec{x}') = b_0}$$

$$q_m = \frac{2\pi}{m}\int_0^\infty dr'\,r'^{m+1}\frac{1}{2\pi}\int_0^{2\pi} d\phi'\,\rho(r',\phi')\,e^{-im\phi'} = \boxed{\frac{1}{m}\int d^2x'\,\rho(x')\,e^{-im\phi'}\,r'^m = q_m}$$

to get accurate info about system at large r (may) only need to calculate a few coef.

Note if ρ is some typical size $\bar{\rho}$ over a region of size L
~~thread~~ and $\rho_n \sim \bar{\rho}$ (it will typically be of order $\bar{\rho}$ or smaller

then

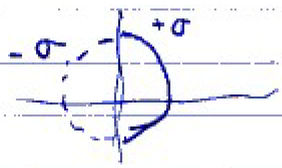
$$q_n \sim \bar{\rho}\, L^{n+2} \qquad \text{(see integral}$$

so n^{th} term in Φ goes as

$$\Phi_n \sim \bar{\rho}\, L^{n+2} \frac{2\pi}{n r^n} \sim \frac{\bar{\rho}\, L^{n+2}}{r^n} \sim \bar{\rho} L^2 \left(\frac{L}{r}\right)^n$$

thus for large r low n's dominant as advertised example — in 3-D consider an ∞ cylindrical shell of radius L. The shell is cut in ½ longitudinally. On the right side of the shell is a uniform surface charge density σ, on the left side of the shell is a uniform surface charge density $-\sigma$. Find the potential Φ and electric field far from the shell

— Note this is really a 2-D problem (no z dependence)



— first calculate the multipole coefficients (sum over charges properly weighted

$$b_0 = -2\int d^2x'\, \rho(x') = 0$$ by sym (same amount of +, - charge

$$q_n = \frac{1}{n}\int d^2x'\, e^{-in\phi}\rho(x')L^n = \frac{L^{n+1}}{n}\left[+\sigma\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-in\phi}\,d\phi - \sigma\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-in\phi}\,d\phi\right]$$

$$= \frac{\sigma L^{n+1}}{n}\left[\frac{1}{-in}\left(e^{-\frac{in\pi}{2}} - e^{+\frac{in\pi}{2}}\right) - \frac{1}{-in}\left(e^{-i\frac{3\pi}{2}n} - e^{+\frac{in\pi}{2}}\right)\right.$$

$$a_n = \frac{2\sigma L^{n+1}}{-i\, n^2}\left[e^{-i\frac{n\pi}{2}} - e^{+i\frac{n\pi}{2}}\right]$$

$$= \frac{2\sigma L^{n+1}}{-i n^2}\left[\left(\cos\left(\tfrac{n\pi}{2}\right) - i\sin\left(\tfrac{n\pi}{2}\right)\right) - \left(\cos\left(\tfrac{n\pi}{2}\right) + i\sin\left(\tfrac{n\pi}{2}\right)\right)\right]$$

$$= \frac{4\sigma L^{n+1}}{n^2}\sin\left(\frac{n\pi}{2}\right)$$

$a_n = 0$ for all even n

$a_n = \frac{4\sigma L^{n+1}}{n^2}(-1)^{\frac{1}{2}(n-1)}$ for all odd n

or setting $n = 2j-1$

$$a_{(2j-1)} = \frac{4\sigma L^{(2j)}}{(2j-1)^2}(-1)^{j-1}$$

Sum up to get Φ for $r>L$

$$\Phi(r,\phi) = \sum_{j>0} r^{-(2j)}\cos((2j-1)\phi)\,\frac{8\sigma L^{(2j)}}{(2j-1)^2}(-1)^{j-1}$$

$$= \sum_{j>0} \frac{8\sigma(-1)^{j-1}}{(2j-1)^2} L\cos((2j-1)\phi)\left(\frac{L}{r}\right)^{2j-1}$$

power series in $\frac{L}{r}$ as advertised

Simple example — Cylinder with $\rho(\vec{r}) = \beta x\, \Theta(R-r)$ where β is a constant and $r = \sqrt{x^2+y^2}$

$b_0 = 0$ by symmetry

$$a_m = \frac{1}{m}\int d^2x'\, \rho(\vec{x}')\, r'^m\, e^{-im\phi}$$

now write $\rho(x')$ in polar $\rho(x') = \beta\, r\cos(\phi)\, \Theta(R-r)$

$$= \frac{\beta}{2} r\left[e^{i\phi} + e^{-i\phi}\right]\Theta(R-r)$$

$$a_m = \frac{1}{m}\int dr'\, r' d\phi\, \tfrac{1}{2}\beta\, r'\, \Theta(R-r')\left(e^{+i\phi'} + e^{-i\phi'}\right)e^{-im\phi}$$

Note ϕ integral $= 0$ unless $m = \pm 1$ in which case it is 2π $\qquad a_m = 0$ for all $m \neq \pm 1$

$$a_1 = 2\pi\left(\tfrac{1}{2}\beta\right)\int_0^R dr'\, r'^2 = \pi\beta\frac{R^3}{3}$$

so

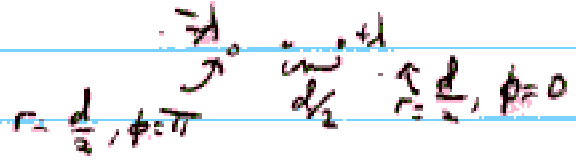
$$\Phi = \frac{1}{r}\left[\pi\beta\frac{R^3}{3}\left(e^{i\phi} + e^{-i\phi}\right)\right]$$

$$= \frac{2\pi\beta R^3}{3r}\cos(\phi)$$

$$E = -\left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial}{\partial\phi}\right]\Phi = \hat{r}\,\frac{2\pi\beta R^3}{3r^2}\cos\phi + \hat{\phi}\,\frac{2\pi\beta R^3}{3r^2}\sin\phi$$

simple example 2

two line charges with charge per unit length oriented in $\hat{z}$ direction are placed at $\pm d/2$



$$\int d^2r' \; \rho(r') \Rightarrow \sum_{i = \text{line charge}} \lambda_i$$

$b_0 = 0$

$$a_n = \frac{1}{n}\int d^2r' \; \rho(r') \, r'^n \, e^{-in\phi}$$

$$= \frac{1}{n}\sum_{i = \text{line charge}} \lambda_i \left(\frac{d}{2}\right)^n e^{-in\phi_i}$$

$$= \frac{1}{n}\left[\lambda\left(\frac{d}{2}\right)^n - \lambda\left(\frac{d}{2}\right)^n e^{-in\pi}\right]$$

$$= \frac{\lambda}{n}\left(\frac{d}{2}\right)^n \left[1 - (-1)^n\right]$$

$a_{-n} = a_n$ as a is real

so $$\Phi = \lambda \sum_n \frac{1}{n} \frac{(d/2)^n}{r^n} \left[1 - (-1)^n\right]\left[e^{in\phi} + e^{-in\phi}\right]$$

$$= \lambda \sum_n \frac{1}{n} \frac{(d/2)^n}{r^n} \left[1 - (-1)^n\right] 2\cos(n\phi)$$

at large distances $n=1$ dominates

$$\Phi = \frac{\lambda d}{r}\cos(\phi)$$

Alternative General expression:

$$\Phi(\vec r) = \int d^2r' \; \rho(\vec r') \left[-2\log\left(\frac{|\vec r - \vec r'|}{r_0}\right)\right]$$

(2-d analog of $\Phi(r) = \int d^3r' \frac{\rho(r')}{|r-r'|}$

proof: first consider a line charge with charge per length λ (in 3d but everything ind. of z)

$\vec E(\vec r)$ by Gauss Law

$$\vec E(\vec r) = \hat r \, E(r)$$

$$\oint \vec E(\vec r)\cdot \hat n \, d^3x = 2\pi r L E(r)$$ concentric cylinder

why none out top?

$$\oint \vec E(r)\cdot \hat n \, d^3x = 4\pi Q^{enclosed} = 4\pi \lambda L$$

$$2\pi r \cancel{L} E(r) = 4\pi \lambda \cancel{L}$$

$$E(r) = \frac{2\lambda}{r}$$

$$\vec E(r) = \frac{2\lambda}{r} \hat r$$

but $\vec{E}(\vec{r}) = -\nabla\Phi$

so $\Phi = -2\lambda \log(r/r_0)$

since then $\vec{\nabla}\Phi = \hat{r}\frac{\partial\Phi}{\partial r} = \frac{2\lambda}{r}$ (no ϕ dependence

superpose different positions

$$\Phi(\vec{r}) = \int d^2r' \, \rho(\vec{r}') \left[-2\log\left(\frac{|\vec{r}-\vec{r}'|}{r_0}\right)\right]$$

Actually we can use this to prove a neat identity

for $|r| > |r'|$ $\quad -2\log\left(\frac{|\vec{r}-\vec{r}'|}{r_0}\right) = -2\log\left(\frac{r}{r_0}\right) + \sum_{n>0}\left(\frac{r'}{r}\right)^n \frac{2}{n}\cos(n(\phi-\phi'))$

$$= 2\log\frac{r}{r_0} + \sum_{n>0}\left(\frac{r'}{r}\right)^n\left[e^{in(\phi-\phi')} + e^{-in(\phi-\phi')}\right]\frac{1}{n}$$

easiest proof plug into general expression and see if we get multipoles

$$\Phi(\vec{r}) = \int d^2r' \, \rho(r')\left[-2\log\frac{|\vec{r}-\vec{r}'|}{r_0}\right]$$

$$= \int d^2r' \, \rho(r')\left\{2\log\frac{r}{r_0} + \sum_{n>0}\left(\frac{r'}{r}\right)^n\left[e^{in(\phi-\phi')} + e^{-in(\phi-\phi')}\right]\frac{1}{n}\right.$$

$$= \int dr' r' d\phi' \, \rho(r')\left\{2\log\frac{r}{r_0} + \sum_{n>0}\left(\frac{r'}{r}\right)^n\left[e^{in(\phi-\phi')} + e^{-in(\phi-\phi')}\right]\frac{1}{n}\right.$$

interchange sum and integral

$$= 2\log\left(\frac{r}{r_0}\right)\int dr' r' \, 2\pi\left(\frac{1}{2\pi}\int_0^{2\pi} d\phi \, \rho(r,\phi)\right) +$$

$$\sum_{n>0}\left[\int dr'(r')^{n+1}\, 2\pi\left(\frac{1}{2\pi}\int_0^{2\pi}\rho(r',\phi)e^{-in\phi}\right)\right]\frac{1}{n}\frac{e^{in\phi}}{r^n}$$

$$+\sum_{n>0}\left[\int_0^\infty dr'(r')^{n+1}\, 2\pi\left(\frac{1}{2\pi}\int_0^{2\pi}\rho(r',\phi')e^{+in\phi'}\right]\frac{e^{-in\phi}}{n r^n}$$

$$= \log\left(\frac{r}{r_0}\right)4\pi\int_0^\infty dr'\,r'\ \rho_0(r)$$

$$+\sum_{n>0}\frac{1}{r^n}\left[e^{in\phi}\frac{2\pi}{n}\int_0^\infty dr'(r')^{n+1}\ \rho_n(\phi)\right]$$

$$+\sum_{n>0}\frac{1}{r^n}\left[e^{-in\phi}\ \frac{2\pi}{n}\int_0^\infty dr'(r')^{n+1}\ \rho_n\phi\right]$$

but this multipole expansion Q.E.D.

example: a "dipole" a line charge of $+\lambda$ at $\vec{r}=\frac{d}{2}\hat{x}$ and $-\lambda$ at $\vec{r}=-\frac{d}{2}\hat{x}$

what is is potential and E field far away? (Leading nonvanishing multipole

exact:

$$\Phi(\vec{r}) = -2\lambda\left[\log\left(\frac{|\vec{r}-\hat{x}\frac{d}{2}|}{r_0}\right)-\log\frac{|\vec{r}+\hat{x}\frac{d}{2}|}{r_0}\right)$$

$$= -2\lambda\log\frac{|\vec{r}-\hat{x}\frac{d}{2}|}{|\vec{r}+\hat{x}\frac{d}{2}|}$$ note indep of r_0 !

for $|r| > d$

[illegible]

$$\Phi(r) = \lambda \left[-2 \log\left(\frac{r}{r_0}\right) + \sum_m \frac{(d/2)^n}{r^n m} \left[e^{in\phi} + e^{-in\phi} \right] \right] \sim r' = 0 \text{ for } \hat{r}' = \frac{d}{2}\hat{x}$$

$$- \lambda \left[-2 \log\left(\frac{r}{r_0}\right) + \sum_m \frac{(d/2)^n}{r^n m} \left[e^{in(\phi+\pi)} + e^{-in(\phi+\pi)} \right] \right] \sim \hat{r}' = -\frac{d}{2}\hat{x}$$

$$= \lambda \sum_{m>0} \frac{(d/2)^n}{r^n m} \left[2\cos(n\phi) \right] \left[1 - (-1)^n \right]$$

only odd terms survive

leading term $m = 1$

$$= \frac{\lambda d}{r} \cos(\phi) \quad + \text{ correction}$$

as seen earlier

Problem with multipole formulation:

$$q_m = \frac{1}{m} \int d^2x' \, \rho(x') \, e^{-i\phi' m} \, r'^m$$

but I measure x' from an arbitrary origin — do the q_m's have any well defined meaning independent of arbitrary origin.

- in general No (but not a problem. pick origin and use it for whole problem a bad choice of origin may make series converge more slowly)

- ~~first~~ lowest non vanishing multipole coef. is independent of origin

proof: consider the following quantity

$$I_n(\phi) \equiv \lim_{r \to \infty} r^n \, \Phi(r, \phi)$$

if I_n is convergent it is clearly independent of origin now use multipole sum

$$I_n(\phi) \equiv \lim_{r\to\infty} r^n \left(b_0 \log\left(\frac{r}{r_0}\right) + \sum_m \frac{1}{r^m}\left(q_m e^{im\phi} + q_m^* e^{-im\phi}\right)\right)$$

- clearly diverges for any n greater than the lowest nonvanishing m

- if lowest nonvanishing $n > 0$

$$I_n = q_n e^{in\phi} + q_n^* e^{-in\phi}$$

but this is ind. of origin
Q.E.D

- if lowest nonvanishing $n = 0$

$$b_0 = -2\int d^2x' \, \rho(\vec{x})$$

clearly ind. of origin
Q.E.D.

example: suppose lowest nonvanishing $m=1$ claim q_1 is ind. of origin.

easy to see:

$$q_1 = \int d^2x' \, \rho(\vec{x}') \, r' e^{-i\phi'}$$

$$= \int d^2x' \, \rho(x') \left[r' \cos\phi' - i r' \sin\phi' \right]$$

$$= \int d^2x' \, \rho(\vec{x}') \left[x' - i y' \right]$$

shift variable in integral

$$\vec{x}\,' = \vec{x}\,'' + \vec{c} \qquad dx' = dx''$$

$\uparrow$ const.

$$q_1 = \int d^2x'' \, \rho(x''+\vec{c}) \left[x'' + c_x + i(y''+c_y) \right]$$

interpret as shift in origin by $\vec{c}$

$$\rho^{new}(x'') = \rho(x''+c)$$

$$q_1 = \int d^2x'' \, \rho^{new}(x'') \, (x'' + i y'')$$

$$+ \int d^2x'' \, \rho^{new}(x'') \, (c_x + i c_y)$$

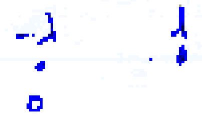
$$= q_1^{new}$$

$$+ \; 0 \quad \left(\text{since } 2\int d^2x'' \rho^{new}(x'') = b_0 = 0\right)$$

so $q_1 = q_1^{new}$

eg

$q_1 = \lambda d$ (from earlier calc.)



$$q_1^{new} = \sum_i \lambda_i e^{-i\phi_i} r_i$$

$$= \lambda d + 0 = \lambda d$$

Q.

Laplace's Equation in 2-D Electrostatics

why bother?

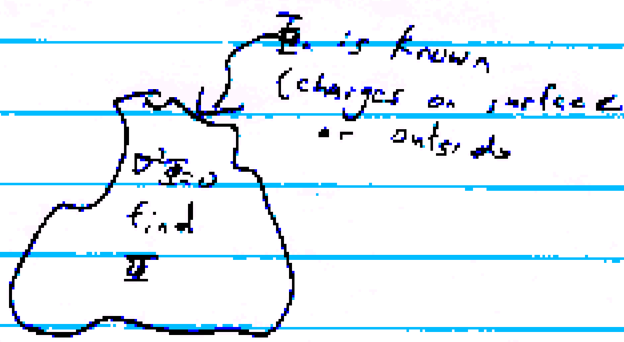
All non-zero solutions of Laplace equations are caused by charges in another region why not just solve Poisson equation over entire system

- ~~sometimes~~ usually we don't know where charges are
- easy fix potentials batteries and conductors
- hard to fix charges

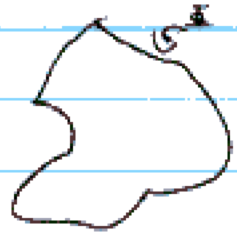
General formulation

$$\nabla^2 \Phi = 0 \quad \text{in a region}$$

$\Phi(\vec{r})$ known on surface bounding region



region can extend to ∞



Φ is known

$\nabla^2\Phi = 0$

find Φ

~~first focus on case where region in 2-d is a solid~~

One method — find a complete set of solution to Laplace equation fix coefficients to match b.c.

example ; $\Phi(r) = b_0 \log\left(\frac{r}{r_0}\right) + \sum_n a_n \frac{e^{in\phi}}{r^n} + b_n e^{-in\phi} r^n + c.c.$

trick is to choose a_n, b_n to match b.c.

easy problem first —

suppose boundary is a circle in 2-d

~~axisymmetric cylinder with Φ ind of z~~

(in 3-d cylindrical shell with Φ ind of z

We can use Fourier methods to find a, b_n

want $\Phi(r,\phi)$ for $r > R$

We know $\Phi(R,\phi)$ (surface)

general form of solution

$$\Phi(r,\phi) = \cancel{b_0 \log(r/r_0)} + \sum_m a_m \frac{e^{im\phi}}{r^m} + \frac{a_m^* e^{-im\phi}}{r^m}$$

if $\Phi \to 0$ @ $r \to \infty$

b_m forms similarly vanish

$$\Phi(R,\phi) = \sum_{m>0} a_m \frac{e^{im\phi}}{R^m} + \frac{a_m^* e^{-im\phi}}{R^m}$$

to find a_m integrate both side

$m' > 0$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-im'\phi}\, \Phi(R,\phi) = \sum_m \frac{a_m}{R^m} \frac{1}{2\pi}\int_0^{2\pi} e^{im\phi} e^{-im'\phi}\, d\phi$$

$$+ \frac{a_m^*}{R^m} \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi} e^{-im'\phi}\, d\phi$$

$$= \sum_m \frac{a_m}{R^m}\, \delta_{mm'} = \frac{a_{m'}}{R^{m'}}$$

$$a_{m'} = R^{m'} \frac{1}{2\pi}\int_0^{2\pi} \Phi(R,\phi)\, e^{-im'\phi}$$

or

$$\boxed{a_m = R^m \frac{1}{2\pi}\int_0^{2\pi} \Phi(R,\phi)\, e^{-im\phi}}$$

we're done

example $\Phi(R,\phi) = V_0 \sin(\phi)$

$$= V_0 \left(\frac{-i}{2}\right)\left[e^{i\phi} - e^{-i\phi}\right]$$

so

$$a_m = R^m \frac{1}{2\pi}\int_0^{2\pi} V_0 \left(\frac{-i}{2}\right)\left[e^{i\phi} - e^{-i\phi}\right] e^{-im\phi}$$

$$= R^m V_0 \left(\frac{-i}{2}\right)\left[\underbrace{\int_0^{2\pi}\left[e^{i\phi}e^{-im\phi} - e^{-i\phi}e^{-im\phi}\right]d\phi}_{\delta_{m1} - \delta_{m-1}}\right]$$

$$a_1 = \frac{-iRV_0}{2}$$

$$a_{-1} = \frac{iRV_0}{2}$$

all other a's zero

for $r > R$

$$\Phi(r,\phi) = \frac{a_1}{r} e^{i\phi} + \frac{a_1^* e^{-i\phi}}{r}$$

$$= \frac{-iRV_0}{2r} e^{i\phi} + \frac{iRV_0}{2r} e^{-i\phi}$$

$$= \frac{R}{r} \sin\phi$$

suppose I want to study region inside $r < R$ know $\Phi(R,\phi)$

general form

$$\Phi(\vec{r}) = \sum_n b_n r^n e^{in\phi} + b_n^* r^n e^{-in\phi}$$

$$\Phi(R,\phi) = \sum_n b_n R^n e^{-in\phi} + b_n^* r^n e^{in\phi}$$

multi by $\frac{1}{2\pi} e^{-in'\phi}$ and integrate

$$R^{n'} b_{n'} = \frac{1}{2\pi} \int d\phi\, e^{-in'\phi}\, \Phi(R,\phi)$$

$$b_n = \frac{1}{R^n} \int d\phi\, e^{-in\phi}\, \Phi(R,\phi)$$

eg $\Phi(R,\phi) = V_0 \sin(\phi)$

same integrals in $r > R$ case

$$b_n = \frac{1}{R^n} \int d\phi\, V_0 \sin(\phi)\, e^{-i\phi n}$$

$$= \frac{V_0}{R^n} \left[\frac{-i}{2} (\delta_{n1} - \delta_{n,-1}) \right]$$

$r < R$

$$\Phi(r, \Phi) = \frac{V_0 r}{R 2} i \left[e^{i\phi} + e^{-i\phi} \right]$$

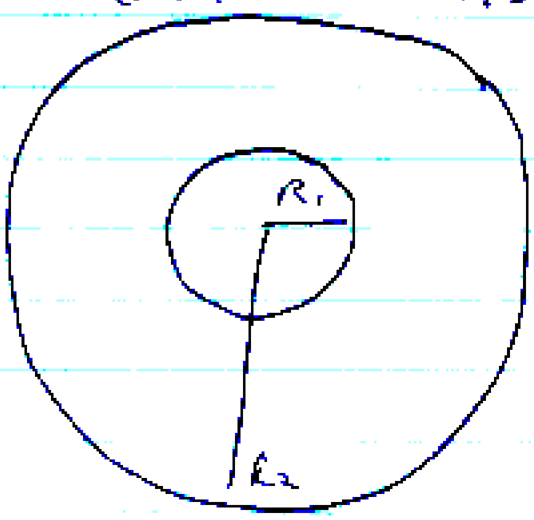
$$= \frac{V_0 r}{R} \sin\phi$$

$$= \frac{V_0 y}{R}$$

const E field inside

$$-\nabla \Phi = -\hat{y} \frac{V_0}{R}$$

Other cases



find Φ inside if Φ on surface is known

$$\Phi(r) = \sum_{n>0}\left(\frac{a_n}{r^n} + b_n r^n\right)e^{in\phi} + c.c.$$

$$+ b_0 \log(r/r_0)$$

b. c. $\Phi(R_1,\phi) \equiv \Phi_0^{(1)}(\phi)$

$\Phi(R_2,\phi) \equiv \Phi_0^{(2)}(\phi)$

so

$$\Phi_0^{(1)}(\phi) = \sum_{n>0}\left(\frac{a_n}{R_1^n} + b_n R_1^n\right)e^{in\phi} + c.c. + b_0 \log\left(\frac{R_1}{R_0}\right)$$

project out part with fixed n

by multiply by $\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-in'\phi}$

$$\Phi^{(1)}_{m'} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-im'\phi}\, \Phi_1(\phi) = \frac{a_{m'}}{R_1^{m'}} + b_{m'} R_1^{m'}$$

for $m' \neq 0$

$$\Phi^{(1)}_0 \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \Phi_1(\phi) = b_0 \log\left(\frac{R_1}{r_0}\right)$$

similarly

$$\Phi^{(2)}(\phi) = \sum_{m \neq 0} \left(\frac{a_m}{R_2^m} + b_m R_2^m\right) e^{im\phi} + b_0 \log\left(\frac{R_2}{r_0}\right)$$

so

$$\Phi^{(2)}_{m'} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{-im'\phi}\, \Phi_2(\phi) = \frac{a_{m'}}{R_2^{m'}} + b_{m'} R_2^{m'}$$

$m' \neq 0$

$$\Phi^{(2)}_0 \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \Phi_2(\phi) = b_0 \log\left(\frac{R_2}{r_0}\right)$$

so we can now extract the a's, b's

example ~~$\Phi_1(\phi)$~~

$\Phi^{(2)}(\phi) = 0$

$\Phi^{(1)}(\phi) = V_0 \cos(\phi)$

$$\Phi_n^{(1)} = \frac{1}{2\pi}\int d\phi\, e^{-in\phi}\, \Phi_1(\phi) = \frac{V_0}{2}\,\delta_{n1}$$

$$\Phi_n^{(2)} = 0$$

so

$n=0$

$$\left.\begin{aligned} b_0 \log\left(\frac{R_2}{r_0}\right) &= 0 \\ b_0 \log\left(\frac{R_1}{r_0}\right) &= 0 \end{aligned}\right\} \quad \text{Solution } b_0 = 0$$

$n=1$

$$\frac{a_1}{R_1} + b_1 R_1 = \frac{V}{2}$$

$$\frac{a_1}{R_2} + b_1 R_2 = 0$$

$$b_1 = -\frac{a_1}{R_2^2}$$

so $\dfrac{a_1}{R_1} - \dfrac{a_1 R_1}{R_2^2} = \dfrac{V_0}{2}$

or $a_1\left(\dfrac{1}{R_1} - \dfrac{R_1}{R_2^2}\right) = \dfrac{V_0}{2}$

$$a_1 = \frac{V_0}{2}\,\frac{1}{\left(\frac{1}{R_1} - \frac{R_1}{R_2^2}\right)} = \frac{V_0}{2}\,\frac{R_1}{1 - \frac{R_1^2}{R_2^2}}$$

$$b_1 = \frac{-a_1}{R_2^2} = \frac{V_0}{2}\,\frac{R_1}{1 - \frac{R_1^2}{R_2^2}}\,\frac{1}{R_2^2} = \frac{V_0}{2}\,\frac{R_1}{R_2^2 - R_1^2}$$

$n > 0$ $\quad a = b = 0$

so

$$\Phi(r,\phi) = \frac{a_1 e^{i\phi}}{r} + b_1 r e^{i\phi} + c.c.$$

$$= \frac{V_0}{2} \frac{R_1}{1-\frac{R_1^2}{R_2^2}} \frac{1}{r} e^{i\phi} + c.c.$$

$$- \frac{V_0}{2} \frac{R_1}{R_2^2-R_1^2} r e^{i\phi} + c.c.$$

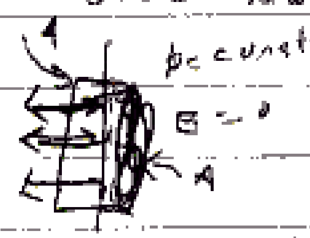
$$= V_0 \cos(\phi) \left[\frac{1}{r} \frac{R_1}{1-\frac{R_1^2}{R_2^2}} - r \frac{R_1}{R_2^2-R_1^2} \right]$$

Find the charge density on surface of Q at R_2 (conductor)

general rule for a const pot. surface

$$\sigma = 4\pi \, E \cdot \hat{n}$$

Gauss law



$$\oint \vec{E} \cdot d\hat{n} = 4\pi Q^{en} \qquad \sigma A = Q^{enc}$$

$$\vec{E} \cdot \hat{n} A = 4\pi \sigma A$$

$$\vec{E}\cdot\hat{n} = -\vec{E}\cdot\hat{r} =$$

$$\vec{E}\cdot\hat{r} = -\frac{\partial\Phi}{\partial r}$$

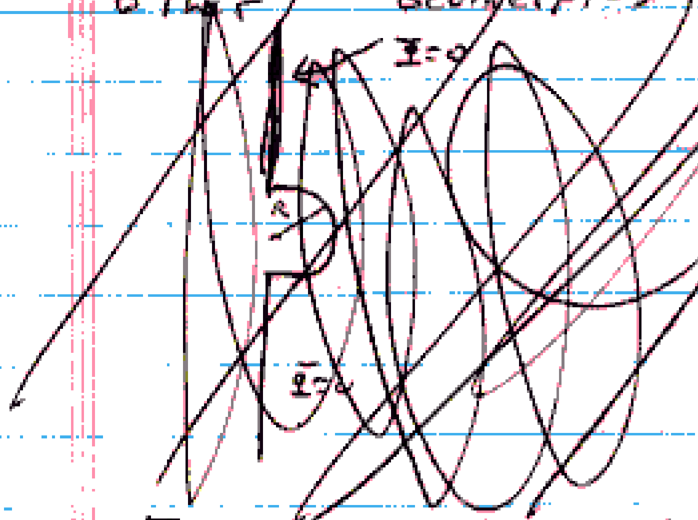
so $$\vec{E}\cdot\hat{n} = +\frac{\partial\Phi}{\partial r}$$

$$= V_0\cos\phi\left[-\frac{1}{r^2}\frac{R_1}{1-\frac{R_1^2}{R_2^2}} - \frac{R_1}{R_2^2-R_1^2}\right]_{r=R_2}$$

$$= V_0\cos\phi\left[-\frac{2R_1}{R_2^2-R_1^2}\right]$$

$$\sigma = V_0\cos\phi\left[\frac{-8\pi R_1}{R_2^2-R_1^2}\right]$$

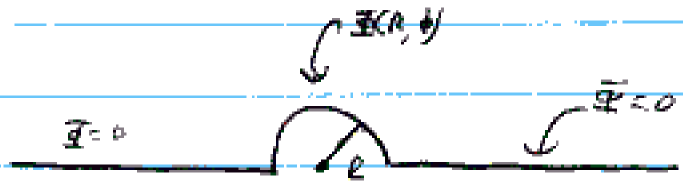
~~other Geometries work for $x>0$~~



$\Phi = 0$ for $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$

-38

other geometries



$\Phi=0$ at $\phi=0,\pi$

work for $y>0$ $0<\phi<\pi$

write general solution using sin & cos

$$\Phi(r,\phi) = b_0 \log(r/r_0)$$

$$+ \sum_m \left[a_m^{(c)} \cos(m\phi) + a_m^{(s)} \sin(m\phi)\right] \frac{1}{r^m}$$

$$+ \cancel{\sum_m \left[b_m^{(c)} \cos(m\phi) + b_m \sin(m\phi)\right] r^m}$$

$= 0$ by $r \to \infty$ b.c.

claim: all $a_m^{(c)}$ terms are zero

since $\cos(m\phi) = \pm 1$ at $\phi = 0, \pi$

$a_m^{(c)} \cos(m\phi) = \pm a_m^{(c)}$ at $\phi=0,\pi$

but $\Phi(r,\phi=0) = \Phi(r,\phi=\pi) = 0$

$$\Phi(r,\phi) = \sum_m \frac{a_m^s \sin(m\phi)}{r^m}$$

trick * imagine the following problem in all space

$\Phi(R,\phi)$

$\Phi(R,-\phi) = -\Phi(R,\phi)$

by sym. $\Phi(r,\phi=0) = \Phi(r,\phi=\pi) = 0$

matches b.c for problem of interest

has exactly the right form

$$a_n^s = R^n \frac{1}{\pi}\int_0^{2\pi} \Phi(R,\phi)\sin(n\phi)$$

generally

so here

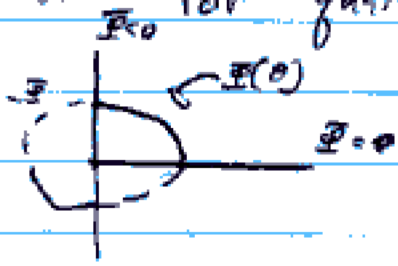
$$a_n^s = R^n \frac{1}{\pi}\left[\int_0^{\pi} \Phi(R,\phi)\sin(n\phi) + \int_{-\pi}^{0} \Phi(R,\phi)\sin(n\phi)\,d\phi\right]$$

let $\phi' = -\phi$

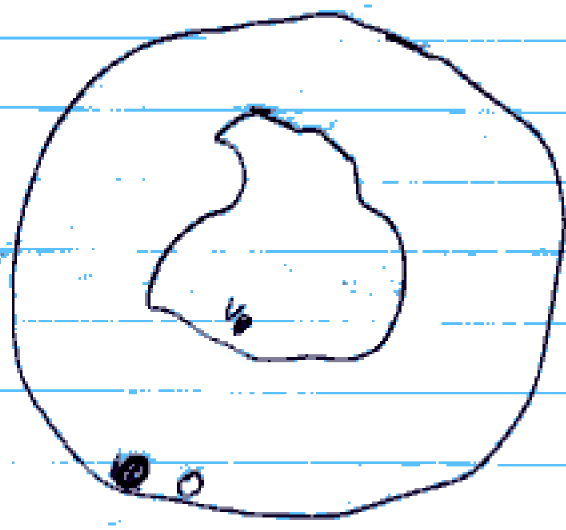
$$= \frac{R^n}{\pi}\left[\int_0^{\pi} \Phi(R,\phi)\sin(n\phi) + \underbrace{\int_{\pi}^{0} \Phi(R,-\phi')\sin(-n\phi')(-d\phi')}_{\underbrace{-\int_0^{\pi}\left(-\Phi(R,\phi')\right)\left(-\sin(n\phi')\right)(-d\phi')}_{+\int_0^{\pi}\Phi(R,\phi')\sin(n\phi')\,d\phi'}}\right]$$

$$= \frac{2R^n}{\pi}\int_0^{\pi} \Phi(R,\phi)\sin(n\phi)$$

what about for quarter circle?



What about radically different geometries?



actually this case is importan
suppose $\Phi = V_0$ on surface 1
$\Phi = 0$ on surface 2
surface 1 + 2 are then conductors in equilibrium

how do I find Φ in region in middle — very important exploit multipole expansion

$$\Phi(r,\phi) = \sum_n a_n r^{-n} e^{in\phi} + b_n r^n e^{in\phi} + c.c.$$
$$+ b_0 \log(r/R_0)$$

to be useful this must converge
how many terms?

claim — "smoothness" of surface sets number of terms needed.

quantify: typical value of

$$S \equiv \frac{1}{R}\frac{dR}{d\phi} \quad \text{dimensionless}$$

if $S = 0$ everywhere 1 term

guess $n_{max} \gg S_{typical}$ to ensure convergence near surface

~~How do I fix the scale~~

Practical test —

keep adding n's until results don't change on scale of accuracy of interest

- How do we fix coefficients — numerically
- We want to match multipole expression to boundary, but... Φ on boundary is a function with an ∞ amount of info so finite # of multipoles can exactly match

trick —
put a set of discrete points on the surfaces and match at these points

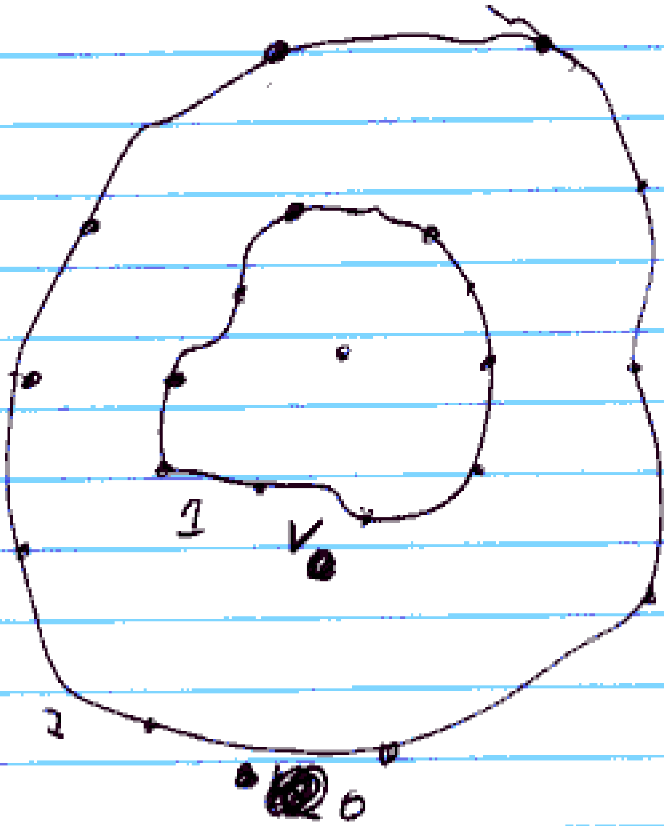
where do we put these points on surface?

- ~~In general evenly around surface~~

— simplest evenly distributed by some criterion

— more accurate put points more densely

in regions of high curvature (those regions are harder to describe with fixed # of moments



Pick N points on each surface (N_1 on inner, N_2 on outer)

- typical point is point i

Surface is specified by

$R_1(\phi)$

$R_2(\phi)$

Thus all we need to specify points is to specify which surface and the angle

$\phi_i^{(1)}, \phi_i^{(2)}$

i^{th} point on surface 1

total # of points = # of free coef to fix

$N_1 + N_2$ = 2 coef per multipole
× 2 real parameters coef
× # of multipoles > 0
+ 2 (2 real coefs for $m=0$)
$= 4\, m_{max} + 2$

b.c. on surface 1 at point i

$$V = \Phi\left(R_1(\phi_i^{(1)}), \phi_i^{(1)}\right)$$

$$= b_0 \log\left(\frac{R_1(\phi_i^{(1)})}{R_0}\right) + \sum_{m>0}\left(a_m \frac{e^{im\phi_i^{(1)}}}{R_1(\phi_i^{(1)})^m} + b_m e^{im\phi_i^{(1)}} R_1(\phi_i^{(1)})^m\right)$$

$+ c.c.$

and on surface 2

$$0 = b_0 \log\left(\frac{R_2(\phi_i^{(2)})}{r_0}\right) + \sum_{m>0}\left(\left(\frac{a_m e^{im\phi_i^{(2)}}}{R_2(\phi_i^{(2)})^m} + b_m e^{im\phi_i^{(2)}} R_2(\phi_i^{(2)})^m\right)\right)$$

there are a total of $N_1 + N_2$ such equations

"just" solve

How — Not by hand!!!

in fact it is not too hard as equations are linear. (Actually the $n=0$ term is not linear as it depends on $b_0 \log\left(\frac{R}{r_0}\right)$ with b_0, r_0 as coef.

trick choose r_0 as an arbitrary # and add a constant a_0
the value of a_0 will depend on r_0
actually earlier we had eliminated a_0 by fixing r_0 to absorb its role and I'm just undoing this)

$$V = b_0 \log\left(\frac{R_1(\phi_i^{(1)})}{\underset{\text{arbt. but fixed}}{r_0}}\right) + a_0 + \sum_{n>0}\left(\frac{a_m e^{in\phi_i^{(1)}}}{R_1(\phi_i^{(1)})^n} + b_m e^{in\phi} R_1(\phi_i^{(1)})^n\right) + c.c$$

$$0 = b_0 \log\left(\frac{R_2(\phi_i^{(2)})}{r_0}\right) + a_0 + \sum_{n>0}\left(\frac{a_m e^{in\phi_i^{(2)}}}{R_1(\phi_i^{(2)})^n} + b_m e^{in\phi} R_2(\phi_i^{(2)})^n + c.c\right)$$

use matrix method to solve

$$\mathcal{C} = \begin{pmatrix} a_0 \\ b_0 \\ a_1 \\ a_1^* \\ b_1 \\ b_1^* \\ a_2 \\ a_2^* \\ b_2 \\ b_2^* \\ \vdots \\ a_{n_{max}} \\ a^*_{n_{max}} \\ b_{n_{max}} \\ b^*_{n_{max}} \end{pmatrix} \qquad V = \begin{pmatrix} V \\ V \\ \vdots \\ V \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} N_1 \\ \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix}\right\} N_2 \end{matrix}$$

write $V = M\,\mathcal{C}$

with

$$M = \begin{array}{c} \\ N_1 \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \\ N_2 \left\{ \begin{array}{c} \\ \\ \end{array} \right. \end{array} \begin{pmatrix} 1 & \log \frac{R_1(\phi_1^{(1)})}{r_0} & \frac{e^{i\phi_1^{(1)}}}{R_1(\phi_1^{(1)})} & \frac{e^{-i\phi_1^{(1)}}}{R_1(\phi_1^{(1)})} & e^{i\phi_1^{(1)}} R_1(\phi_1^{(1)}) & e^{-i\phi_1'} R_1(\phi^{(1)}) & \frac{e^{2i\phi_1^{(1)}}}{R_1(\phi_1^{(1)})^2} & \cdots \\ 1 & \log \frac{R_1(\phi_2^{(1)})}{r_0} & \frac{e^{i\phi_2^{(1)}}}{R_1(\phi_1)} & \cdots & & & & \\ \vdots & & & & & & & \\ 1 & \log \frac{R_1(\phi_{N_1}^{(1)})}{r_0} & \frac{e^{i\phi_{N_1}^{(1)}}}{R_1(\phi_{N_1}^{(1)})} & \cdots & & & & \\ 1 & \log \frac{R_2 \phi_1^{(2)}}{r_0} & \frac{e^{i\phi_1^{(2)}}}{R_2(\phi_1^{(2)})} & \cdots & & & & \\ \vdots & & & & & & & \end{pmatrix}$$

$(N_1+N_2) \times (N_1+N_2)$ matrix

or $(4 n_{max}+2) \times (4 n_{max}+2)$ matrix

to solve for the coef.

$$V = M C$$

so invert matrix and mul

$$M^{-1} V = M^{-1} M C$$

$$C = M^{-1} V$$

trick is invert M
(Not by hand!!)

Concrete case —

- surface 2 to a cylinder of radius R_2
- surface 1 is an ellipse with major axis α and minor axis β

small eccentricity $\frac{\alpha-\beta}{\alpha} << 1$
this ensure only a few multipoles neede

$$R_2(\phi^{(2)}) = R_2$$

$$R_1(\phi^{(1)}) = \sqrt{\alpha^2 \cos^2(\phi^{(1)}) + \beta^2 \sin^2(\phi)}$$

claim by sym $a_m = a_{-m}$ all a's real
since $R_1(-\phi) = R_1(\phi)$ for all ϕ
and $R_2(-\phi) = R_2(\phi)$

then $\Phi(r, \phi) = \Phi(r, -\phi)$ but this means only cos term

2 coef's per multipole

claim by sym $a_m = 0$ for all odd m

$$R_1(\phi+\pi) = R_1(\phi)$$
$$R_2(\phi+\pi) = R_2(\phi)$$

so $\quad \Phi(r,\phi) = \Phi(r, \phi+\pi)$

but if I write

$$\Phi(r,\phi) = \sum_n \left(\frac{2a_n}{r^n} + 2b_n r^n\right)\cos(\phi n) + b_0 \log\left(\frac{r}{r_0}\right) + a_0$$

we see only n even terms have this property

so for this case

$$C = \begin{pmatrix} a_0 \\ b_0 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \\ \vdots \\ \vdots \end{pmatrix} \qquad V = \begin{pmatrix} v \\ v \\ \vdots \\ v \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & \log\left(\frac{R(\phi)}{r_0}\right) & \frac{2\cos(2\phi_1^{(1)})}{R_1^2(\phi_1^{(1)})} & 2\cos(2\phi_1^{(1)})R_1^2(\phi_1^{(1)}) & \frac{2\cos(4\phi_1^{(1)})}{R_1(\phi_1^{(1)})^4} & 2\cos(\phi_1^{(1)})R_1^4 \\ & & \cdots & & & \end{pmatrix}$$

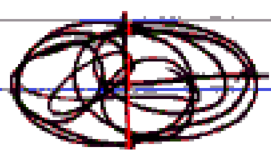
in choosing points, it is sufficient to choose points in 1^{st} quadrant

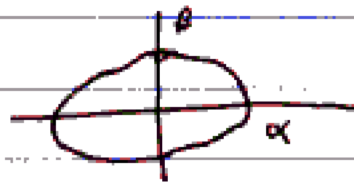
m real and even map it onto all 4

suppose we work up to $m_{max} = 2$

valid iff $\frac{\alpha - \beta}{\alpha + \beta} \ll 1$

4 real coefs — 4 coefs





I'll pick points at $\phi = 0, \frac{\pi}{2}$ (why not?) on both surfaces

pick $r_0 = R_2$ (why not?)

$\phi_1^{(1)} = 0 \quad \phi_1^{(2)} = 0 \qquad R_1(\phi_1^{(1)}) = \alpha \qquad \cos(2\phi_1^{(1)}) = 1$

$\phi_2^{(1)} = \pi/2 \quad \phi_2^{(2)} = 0 \qquad R_1(\phi_2^{(1)}) = \beta \qquad \cos(2\phi_1^{(1)}) = -1$

$$M = \begin{pmatrix} 1 & \log\left(\frac{\alpha}{R_2}\right) & \frac{2}{\alpha^2} & 2\alpha^2 \\ 1 & \log\left(\frac{\beta}{R_2}\right) & -\frac{2}{\alpha^2} & -2\alpha^2 \\ 1 & 0 & \frac{2}{R^2} & 2R^2 \\ 1 & 0 & -\frac{2}{R^2} & 2R^2 \end{pmatrix}$$

Now do numerics!

— Now Let's change directions to 3-D electrostatics

— work by analogy to 2-D case

— keep things simple assume azimuthal symmetry

i.e. ϕ independence

— work in ~~polar~~ spherical coordinates

$$\nabla^2 \Phi = -4\pi \rho$$

$$\rho(\vec{x}) = \rho(r, \theta) \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$\Phi(\vec{x}) = \Phi(r, \theta)$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

look up ∇^2 in ~~[illegible]~~ spherical coordinates or work out

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta}$$
$$+ \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Here I will focus on Laplace equation

$\nabla^2 \Phi = 0$

by hypothesis $\Phi(\vec{x}) = \Phi(r,\theta)$ so last term does not contribute

I'll assume that we can form a complete set of solutions in "separable" form

$\Phi(r,\theta) = R(r)\Theta(\theta)$

(Note ~~our~~ 2-d solutions were of analogus form $r^n e^{im\phi}$)

Next some ugly algebra to get this to work we must have

53

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\quad\Theta(\theta) = \text{const}\ \Theta(\theta)$$

$$= -\ell(\ell+1)\Theta(\theta)$$

$\uparrow$ ind of r

Since then

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r)\Theta(\theta) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}R(r)\Theta(\theta)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r)\Theta(\theta)\ \cancel{\text{and}}\ \frac{-\ell(\ell+1)R(r)}{r^2}\Theta(\theta) = 0$$

$$= \left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r}\right]R(r) = 0$$

$\uparrow$ ind of θ

in that case

$R(r)$ is either $\frac{a_\ell}{r^{\ell+1}}$ or $b_\ell r^\ell$
with a_ℓ, b_ℓ
const.

proof:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\quad r^\alpha = \frac{1}{r^2}\frac{\partial}{\partial r}\alpha r^{\alpha+1} = \frac{\alpha(\alpha+1)}{r^2}r^\alpha$$

so if

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right]r^\alpha = 0$$

then $\alpha(\alpha+1) = \ell(\ell+1)$

works if $\alpha = \ell$

or if $\alpha = -(\ell+1)$

since $\alpha(\alpha+1) = -(\ell+1)(-(\ell+1)+1) = -(\ell+1)(-\ell) = \ell(\ell+1)$

so

$r^{\ell}, \; r^{-(\ell+1)}$ solutions

if

$\Theta(\theta)$ satisfies

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \ell(\ell+1)\right]\Theta(\theta) = 0$$

trick — make change of variable

$x = \cos\theta$ (note not cartesian x)

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = \sin\theta\frac{d}{dx}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} = \frac{d}{dx}\sin^2\theta\frac{d}{dx} = \frac{d}{dx}(1-x^2)\frac{d}{dx}$$

so equation becomes

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + \ell(\ell+1)\right]\Theta = 0$$

But this is a famous d.f. eq

Legendres equation

solutions with integer $l \geq 0$
are the Legendre Polynomials $P_l(x)$

claim — $P_l(x)$ form a complete, orthogonal basis on interval from -1 to 1

so general solution to Laplace eq. for axial symmetric 3-D problem

$\nabla^2 \Phi(r, \theta) = 0$

real

$$\boxed{\Phi(r,\theta) = \sum_{l \geq 0} \left(\frac{a_l}{r^{l+1}} + b_l r^l \right) P_l(\cos\theta)}$$

compare with 2-d

$$\Phi(r,\phi) = \sum_{m>0} \left(\frac{a_m}{r^m} + b_m r^m \right) e^{im\phi} + c.c. + b_0 \log\left(\frac{r}{r_0}\right) \cdot$$

similar structure

what about these P_ℓ

$P_\ell(x)$ is a polynomial in x

State without proof.

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \qquad \text{Rodrigues formula}$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

convention:

$P_\ell(1) = 1$

Factoid

$P_\ell(-x) = (-1)^\ell P_\ell(x)$

check that these solve equation $\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + \ell(\ell+1)\right]P_\ell$

$\ell=0$ $\quad \left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + 0\right] 1 = 0$ ✓

$\ell=1$ $\quad \left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + 2\right] x = \frac{d}{dx}\left[1-x^2\right] + 2x = -2x+2x = 0$ ✓

$\ell=2$ $\quad \left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + 6\right]\frac{1}{2}(3x^2-1) = \frac{d}{dx}(1-x^2)(3x) + 3(3x^2-1) = (3-9x^2)+9x^2-3 = 0$ ✓

key feature i orthogonal in interval -1 to 1

state w/o proof

$$\boxed{\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)\, dx = \delta_{\ell\ell'} \frac{2}{(2\ell+1)}}$$

$$\int_0^\pi P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)\, \sin\theta\, d\theta = \delta_{\ell\ell'} \frac{2}{2\ell+1}$$

suppose I have

$$f(\cos\theta) = \sum_\ell f_\ell\, P_\ell(\cos\theta)$$

then

$$\int_0^\pi P_{\ell'}(\cos\theta)\, f(\cos\theta)\, \sin\theta\, d\theta = \sum_\ell f_\ell \int d\theta \sin\theta\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)$$

$$= \sum_\ell f_\ell\, \delta_{\ell\ell'} \frac{2}{(2\ell+1)}$$

$$= f_{\ell'} \frac{2}{(2\ell'+1)}$$

so

$$\boxed{f_{\ell'} = \frac{2\ell'+1}{2} \int_0^\pi P_{\ell'}(\cos\theta)\, f(\cos\theta)\, \sin\theta\, d\theta}$$

exploit to solve Laplace eq with boundary condition

eg. suppose we know $\Phi(\theta)$ on surface of sphere (easy case analog of circle / cylinder case in 2-d

$\nabla^2 \Phi = 0$ for $r > R$ and for $r < R$

. for $r > R$

$$\Phi(r, \Phi) = \sum_{l \geq 0} \left(\frac{a_l}{r^{l+1}} + \cancel{b_l r^l} \right) P_l(\cos\theta)$$

why?

Now $\Phi(\theta)$ on surface is $\Phi(r = R, \theta)$

$$\Phi(\theta) = \sum_{l \geq 0} \frac{a_l}{R^{l+1}} P_l(\cos\theta) \equiv \sum_{l \geq 0} f_l \, P_l(\cos\theta)$$

with $f_l = \frac{a_l}{R^{l+1}}$

to pick out a_l's same trick as before

so $f_l = \frac{2l+1}{2} \int_0^\pi P_l(\cos\theta) \Phi(\theta) \, d\theta$

and $$\boxed{a_l = R^{l+1} \left(\frac{2l+1}{2} \right) \int_0^\pi P_l(\cos\theta) \, \Phi(\theta) \sin\theta \, d\theta}$$

for $r < R$

$$\Phi(r, \theta) = \sum_{l \geq 0} \left(\cancel{\frac{a_l}{r^{l+1}}} + b_l r^l \right) P_l(\cos\theta)$$

$$\Phi(R, \theta) = \Phi(\theta) = \sum_{l \geq 0} b_l R^l P_l(\cos\theta)$$

of form $= \sum_{l \geq 0} f_l P_l(\cos\theta)$

with $f_l = b_l R^l$

but $f_l = \int_0^\pi P_l(\cos\theta)\, \Phi(\theta) \sin\theta \, d\theta$

so

$$\boxed{b_l = \frac{1}{R^l} \int_0^\pi P_l(\cos\theta)\, \Phi(\theta) \sin\theta \, d\theta}$$

example suppose

$$\Phi(R,\theta) = V_0 \cos\theta$$
$$= V_0 P_1(\cos\theta)$$

so only $r=1$ contribute

so for $r > R$ only $l=1$ means

$$\Phi(r,\theta) = \frac{q_1}{r^2} P_1(\cos\theta)$$

match at $r = R$

$$\Phi(R,\theta) = V_0 P_1(\cos\theta) = \frac{q_1}{R^2} P_1(\cos\theta)$$

so $q_1 = R^2 V_0$

$$\Phi = \frac{R^2 V_0}{r^2} P_1 \cos\theta$$

suppose I didn't recognize that

$V_0 \cos\theta = V_0 P_1(\cos\theta)$

$$q_l = R^{l+1}\left(\frac{2l+1}{2}\right)\int_0^\pi P_l(\cos\theta)\,\Phi(\theta)\sin\theta\,d\theta$$

$$= R^{(l+1)}\left(\frac{2l+1}{2}\right)\int_0^\pi P_l(\cos\theta)V_0\cos\theta\sin\theta\,d\theta$$

$$= V_0 R^{l+1}\left(\frac{2l+1}{2}\right)\int P_l(x)\,x\,dx$$

Now $\int_{-1}^{1} P_\ell(x) x \, dx = \frac{2}{3}$ for $\ell = 1$
$= 0$ otherwise } explicit

or $a_\ell = V_0 R^2 \delta_{\ell,0}$

and

$$\Phi(r,\theta) = \sum_\ell \frac{a_\ell}{r^{\ell+1}} P_\ell(\cos\theta) = \frac{R^2 V_0}{r^2} P_1(\cos\theta) = \frac{R^2 V_0}{r^2} \cos\theta$$

Now suppose

$\Phi(R,\theta) = V_0 \cos\theta$

on surface of sphere

$\nabla^2 \Phi = 0$ for $r < R$

what is $\Phi(r)$ inside

$\Phi(R,\theta) = V_0 P_1(\cos\theta)$

General solution

$$\Phi(r,\theta) = \sum_\ell \left(\frac{a_\ell}{r^{\ell+1}} + b_\ell r^\ell \right) P_\ell(\cos\theta)$$

b.c. at $r=0$ Φ is sensible $a_\ell=0$

$$\Phi(r,\theta) = \sum_\ell b_\ell r^\ell P_\ell(\cos\theta)$$

match at $r=R$

$$\Phi(R,\theta) = \sum_\ell b_\ell R^\ell P_\ell(\cos\theta)$$

$$= V_0 P_1(\cos\theta)$$

$\therefore$ $R b_1 = V_0$; $b_1 = \frac{V_0}{R}$ $b_\ell = 0$ for $\ell \neq 1$

$$\Phi(r,\theta) = \frac{V_0}{R} r P_1(\cos\theta)$$

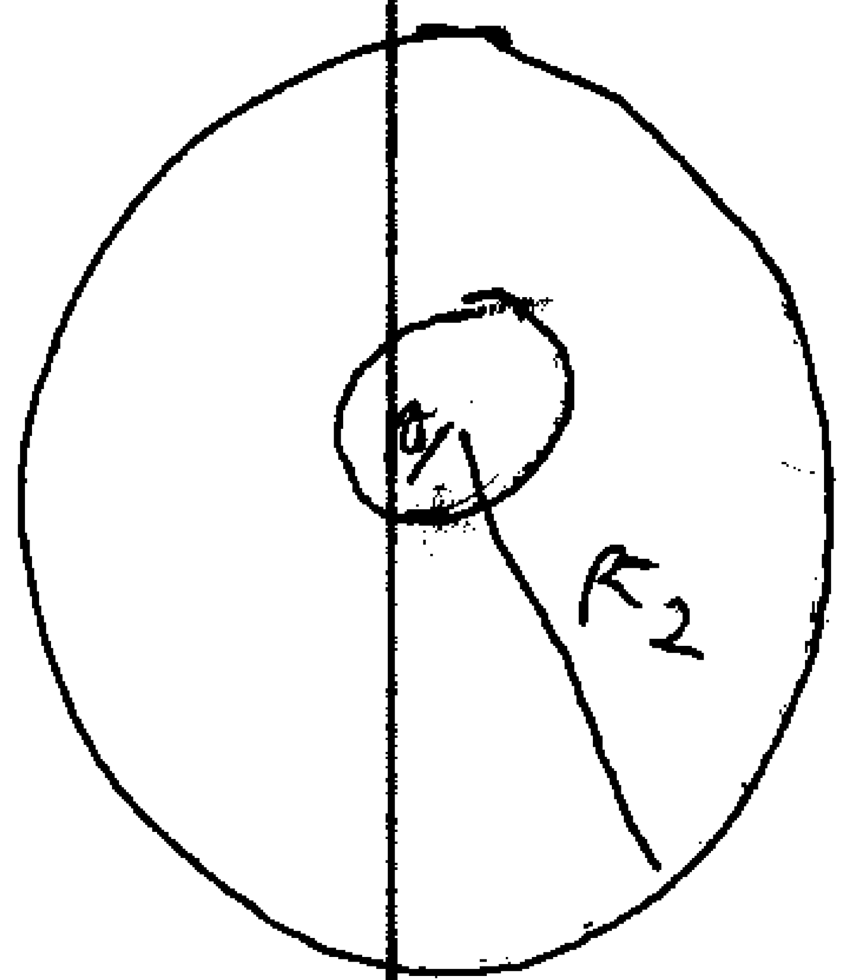
$$= \frac{V_0}{R} r \cos\theta$$

$$= \frac{V_0}{R} z$$

$$\vec{E} = -\nabla\Phi = -\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\frac{V_0}{R} z\right)$$

$$= -\hat{z}\frac{V_0}{R}$$

Two Concentric Spheres



$\nabla^2 \Phi = 0$ in intermediate zone

$$\Phi(r,\theta) = \sum_{l} \left(\frac{a_l}{r^{l+1}} + b_l r^l \right) P_l(\cos\theta)$$

Match on surface 1

$$\Phi(R_1, \phi) = \sum_{l'} \left(\frac{a_{l'}}{R_1^{l'+1}} + b_{l'} R_1^{l'} \right) P_{l'}(\cos\theta)$$

mult both sides by $P_l(\cos\theta)$ and integrate

$$\int_0^\pi P_l(\cos\theta)\, \Phi(R_1,\theta) \sin\theta \, d\theta = \frac{2l+1}{2} \left(\frac{a_l}{R_1^{l+1}} + b_l R_1^l \right)$$

$$\Phi_2(R_2,\theta) = \sum_l^\infty \left(\frac{a_l}{R^{l+1}} + b_l R^l\right) P_l(\cos\theta)$$

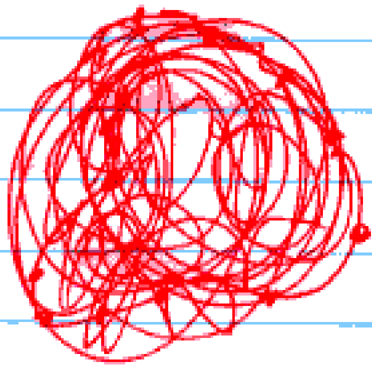
so

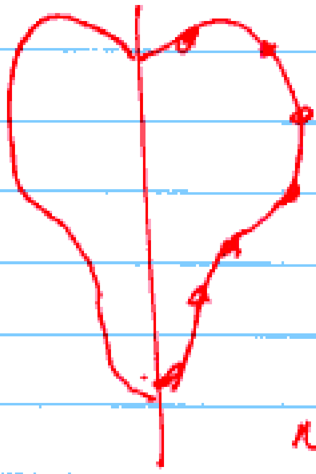
$$\left(\frac{a_l}{R_2^{l+1}} + b_l R_2^l\right) = \frac{2l+1}{2}\int_0^\pi P_l(\cos\theta)\,\Phi_2(\theta)\sin\theta\, d\theta$$

What if boundary not sphere?

Numerical. how?

suppose $\Phi = V$ on surface $\Phi = 0$ at $r \to \infty$





N points

$N = l_{max} + 1$

$$\Phi(r,\theta) = \sum_0^l \frac{a_l}{r^{l+1}} P_l(\cos\theta)$$

match on N points

$R(\theta)$ defines surface

$$V = \Phi(R(\theta_i),\theta_i) = \sum_l \frac{a_l}{R(\theta)^{l+1}} P_l(\cos\theta)$$

Solve as a matrix

$$\begin{pmatrix} v \\ v \\ v \\ v \\ \vdots \\ v \end{pmatrix} = \begin{pmatrix} \frac{P_0(\cos\theta_1)}{R(\theta_1)} & \frac{P_1(\cos\theta_1)}{R^2(\theta_1)} & \frac{P_2(\cos\theta_1)}{R^3(\theta_1)} & \cdots & \frac{P_{\ell_{max}}(\cos\theta_1)}{R^{\ell_{max}+1}} \\ \frac{P_0(\cos\theta_2)}{R(\theta_2)} & \frac{P_1(\cos\theta_2)}{R^2(\theta_2)} & \frac{P_2(\cos\theta_2)}{R^3(\theta_2)} & \cdots & \\ & \vdots & & & \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{\ell_{max}} \end{pmatrix}$$

$$\vec{v} \qquad \overleftrightarrow{M} \qquad \vec{A}$$

$$\overleftrightarrow{M}\vec{A} = \vec{v}$$

$$\vec{A} = \overleftrightarrow{M}^{-1}\vec{v}$$

inverting matrix solves problem

How many ℓ's do we need?

More pts needed the more structure in shape what about Poisson equation

$$\nabla^2 \Phi = -4\pi\rho$$

$\Phi(r,\theta) = \sum_l \Phi_l(r) P_l(\cos\theta)$

$\rho(r,\theta) = \sum_l \rho_l(r) P_l(\cos\theta)$

modeled on 2-d experience

where $\rho_l = \frac{2l+1}{2} \int_0^\pi P_l(\cos\theta)\, \rho(r,\theta) \sin\theta\, d\theta$

recall

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \cancel{\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}$$

$$\nabla^2 \Phi_l(r) P_l(\cos\theta) = \left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] \Phi_l(r) P_l(\cos\theta)$$

so

$$\nabla^2 \Phi = \sum_l \left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] \Phi_l(r) P_l(\cos\theta)$$

$$-4\pi\rho = -4\pi \sum_l \rho_l(r) P_l(\cos\theta)$$

Must be true l by l

Why?

- each P_l has different angular dependence only way true for all angles is if true for each l
- mult both sides by $P_{l'}(\cos\theta)$ and integrate $\int d\theta \sin\theta$
  pick out single l

ergo $\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right]\Phi_\ell = -4\pi\, \rho_\ell(r)$

OOC.

how to solve:

inspired guess..

$$\Phi_\ell(r) = \frac{A}{r^{\ell+1}}\int_0^r dr'\, \rho_\ell(r')\, r'^{\ell+2} + B r^2 \int_r^\infty dr' \frac{\rho_\ell(r')}{r'^{(\ell-1)}}$$

A & B are consts.

why? dimensionally

$\Phi \sim \frac{Q}{r}$ $\quad \rho \sim \frac{Q}{r^2}$

plug in to fix A & B

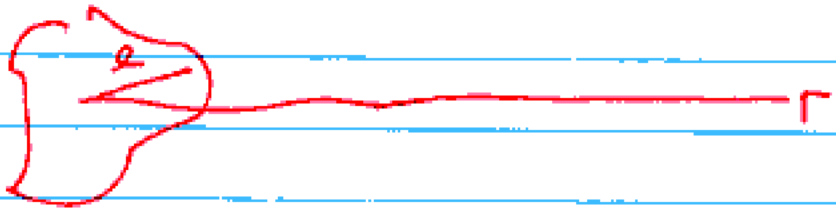
get $A = -B$

$A = \frac{4\pi}{2\ell+1}$

so

$$\Phi_\ell(r) = \frac{4\pi}{2\ell+1}\frac{1}{r^{\ell+1}}\int_0^r dr'\, \rho_\ell(r')\, r'^{\ell+2} - \frac{4\pi}{2\ell+1} r^\ell \int_r^\infty dr' \frac{\rho_\ell(r')}{r'^{(\ell-1)}}$$

suppose charge distribution is localized



and $r \gg \ell$

for $r > \ell$ we have solution of Poisson equation = solution of Laplace

$$\Phi(r,\theta) = \sum_\ell \frac{q_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

why ∞ ?

$\int$ from ℓ to ∞ doesn't matter

so

$$q_\ell = \frac{4\pi}{2\ell+1}\int_0^\infty dr'\, \rho_\ell(r')\, r'^{(\ell+2)}$$

$$= \frac{4\pi}{2\ell+1}\int_0^\infty dr'\, r'^2\, r'^\ell \left(\frac{2\ell+1}{2}\right)\int_0^\pi P_\ell(\cos\theta)\,\rho(r,\theta)\sin\theta\, d\theta$$

$$= 2\pi \int_0^\infty dr'\, r'^2 \sin\theta \int_0^\pi d\theta\, \rho(r,\theta)\, P_\ell(\cos\theta)\, r'^\ell$$

$$= \underbrace{\int_0^\infty dr' \int_0^\pi d\theta \int_0^{2\pi} d\phi\; r'^2 \sin^2\theta}\; \rho(r',\theta)\, P_\ell(\cos\theta)\, r'^\ell$$

$$= \int_{vol} d^3x'\, \rho(r',\theta)\, r'^\ell\, P_\ell(\cos\theta)$$

eg suppose $\rho(r,\theta) = z\,\Theta(R-r)\,\alpha_0$

what are q_ℓ's

$$q_\ell = \int d^3x \;\; \alpha_0 \, z \, \Theta(R-r) \;\; P_\ell(\cos\theta)\, r^\ell$$

$$= \alpha_0 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{R} dr \, r^2 \;\; r\cos\theta \;\; r^\ell \;\; P_\ell(\cos\theta)$$

$$= \alpha_0 2\pi \int_0^{\pi} \sin\theta \, d\theta \;\; P_\ell(\cos\theta) \cos\theta \int_0^{R} dr \;\; r^{\ell+3}$$

$$= \alpha_0 \;\; 2\pi \left[\int_0^{\pi} \sin\theta \, d\theta \;\; P_\ell(\cos\theta) P_1(\cos\theta) \right] \frac{R^{\ell+4}}{\ell+4}$$

$$= \alpha_0 \;\; 2\pi \left(\frac{2}{2\ell+1} \right) \delta_{\ell 1} \;\; \frac{R^{\ell+4}}{\ell+4}$$

$$= \alpha_0 \, 2\pi \left(\frac{2}{3} \right) \frac{R^5}{5} \;\; \delta_{\ell 1}$$

so

$$\Phi(r,\theta) = \frac{\alpha_0 \, 4\pi}{15} \frac{R^5}{r^2} \cos\theta$$