Relativity —

- treatment will be formal
- stress transformation properties

Principle of Relativity —
Laws of physics are same in all inertial frames

what is inertial frame — set of kinematical variables $(\vec{x}, t)$ such that a particle with no force on it experiences no acceleration

Old idea — goes back to Galileo

in Newtonian physics

$$\vec{x}' = \vec{x} - \vec{v}t$$

transforms from one frame to another

clearly satisfies relativity principle

$$\vec{F} = ma = m\ddot{\vec{x}}$$

in prime frame

$$\vec{F}' = m\ddot{\vec{x}}' = m\frac{d^2(\vec{x} - \vec{v}t)}{dt^2} = m\ddot{\vec{x}} = \vec{F}$$

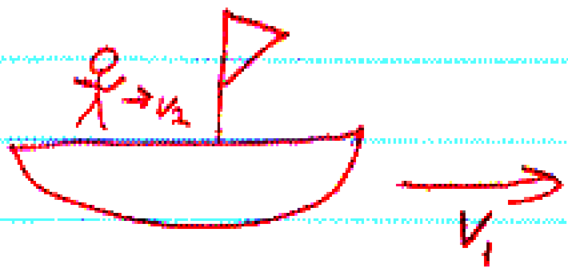
force is same in all frames

laws of physics expressed thru forces

i.e. law of physics same in all frames

— useful: do calculations in simplest frame
e.g. collisions in center of mass frame

consequence of Galilean Relativity — addition of velocity



$\vec{V}_{total} = \vec{v}_1 + \vec{v}_2$

proof

$\vec{x}_1 = \vec{x}_0 + \vec{v}_1 t$

$\vec{x}_2 = \vec{x}_1 + \vec{v}_2 t = \vec{x}_0 + \vec{v}_1 t + \vec{v}_2 t = \vec{x}_0 + \underbrace{(\vec{v}_1 + \vec{v}_2)}_{V_{total}} t$

~~[illegible]~~

2nd Postulate of Einstein Relativity: speed of light is the same in all frames

why? i) Maxwell's equations — give speed of light as c but don't specify frame. If Maxwell's equations are laws of physics then by principle of relativity c is same in all frame

ii) Michaelson - Morely experiment

2nd postulate of relativity incompatible with Galilean transformation

suppose frame 0 and 1 differ by c in $\hat{x}$ direction, and frame 2 and 1 also differ by c

$$\vec{x}_1 = \vec{x}_0 - \hat{x}ct$$

and

$$\vec{x}_2 = \vec{x}_1 - \hat{x}ct = \vec{x}_0 - 2ct$$

but this says speed of light is 2c measured in frame 2 (with light source in frame conclusion : Galilean transformations fail. New transformations needed

Word about units — factors of c are a pain in the ass! why not just set c =

problem — how can I set c to 1 when $c = 2.99 \times 10^8$ m/s ?

solution — don't use meters and s
Pick units with $c = 1$

e.g. t ~ secs   length ~ light-sec   1 light-sec = 2.99×

to find transformation consider following thought experiment — at $t=0, \vec{x}=0$ in some frame flash a light for an instant. At a later time t the light passes thru a sphere :

$$\sqrt{x^2+y^2+z^2} = ct \quad \overset{c \text{ units}}{=} t$$

or

$$t^2 - (x^2 + y^2 + z^2) = 0$$

Now suppose we did this in another inertial frame lined up so that the flash occurred at $t' = 0$, $\vec{x}' = 0$

if speed of light is same in all frames

$$t'^2 - (x'^2 + y'^2 + z'^2) = 0$$ describes sphere that light reaches in this frame

thus

$$t^2 - x^2 - y^2 - z^2 = t'^2 - x'^2 - y'^2 - z'^2$$

what transformation of x, t satisfies this

- rotations (boring!)
- Lorentz boosts

for simplicity I'll do this in x direction

$$\left.\begin{aligned} t' &= \gamma t - \beta\gamma x \\ x' &= -\beta\gamma t + \gamma x \\ y' &= y \\ z' &= z \end{aligned}\right\} \quad \beta = v, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

it is straightforward to show that with this transformation

$$t'^2 - x'^2 - y'^2 - z'^2 = t^2 - x^2 - y^2 - z^2$$ Prove for (homework)

Factoid: limit $v \ll 1$ (i.e. $v \ll c$)

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-v^2}} = 1 + \tfrac{1}{2}\beta^2 + \mathcal{O}(\beta^4)$$

$$= 1 + \mathcal{O}(\beta^2)$$

$$\beta\gamma = v + \mathcal{O}(v^3)$$

so up to order v in an expansion

$x' = x - vt$ is same as nonrelativistic Galilean trans!

Einstein relativity → Galilean reality for $v \ll 1$

Easy way to represent Lorentz trans — Matrix, 4-vector

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

x'^μ $\qquad$ Λ matrix representing Lorentz boost $\qquad$ call this x^μ ← notation indicating 4-

$$\vec{X}'^{4} = \overleftrightarrow{\Lambda}\,\vec{X}'$$

Word on notation — I will put superscript 4 to indicate 4 vectors this is not standard. Usually this is done by using Greek indices but to do this I need the full co- contra variant formalism

Lorentz transformations act on points in space-time. Space time points label events

space time 4 dimensional — need 4 numbers to specify an event
3 numbers to specify where, 1 number to specify when

Lorentz trans

$$\vec{x}'^{4} = \Lambda \, \vec{x}^{4}$$

Consider two events each labeled by a 4vector in space time

$$\vec{x}^{4}_{a}, \quad \vec{x}^{4}_{b}$$

$$\Delta \vec{x}^{4} = \vec{x}^{4}_{b} - \vec{x}^{4}_{a}$$

in primed frame:

$$\vec{x}^{4\prime}_{b} = \overleftrightarrow{\Lambda} \, x^{4}_{b}$$

$$\vec{x}^{4\prime}_{a} = \overleftrightarrow{\Lambda} \, x^{4}_{a}$$

so

$$\Delta \vec{x}^{4\prime} = \vec{x}^{4\prime}_{b} - \vec{x}^{4\prime}_{a} = \overleftrightarrow{\Lambda} \vec{x}^{4}_{b} - \overleftrightarrow{\Lambda} \vec{x}^{4}_{a}$$

$$= \overleftrightarrow{\Lambda} (\vec{x}^{4}_{b} - x^{4}_{a})$$

$$= \overleftrightarrow{\Lambda} \Delta x$$

$$\begin{pmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Since $\Delta\vec{x}^4$ transforms same way as $\vec{x}^4$

$$\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Quantity $\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ is same in all frames: it is Lorentz invariant

define $\tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$

For any two events we can define τ^2
three cases:

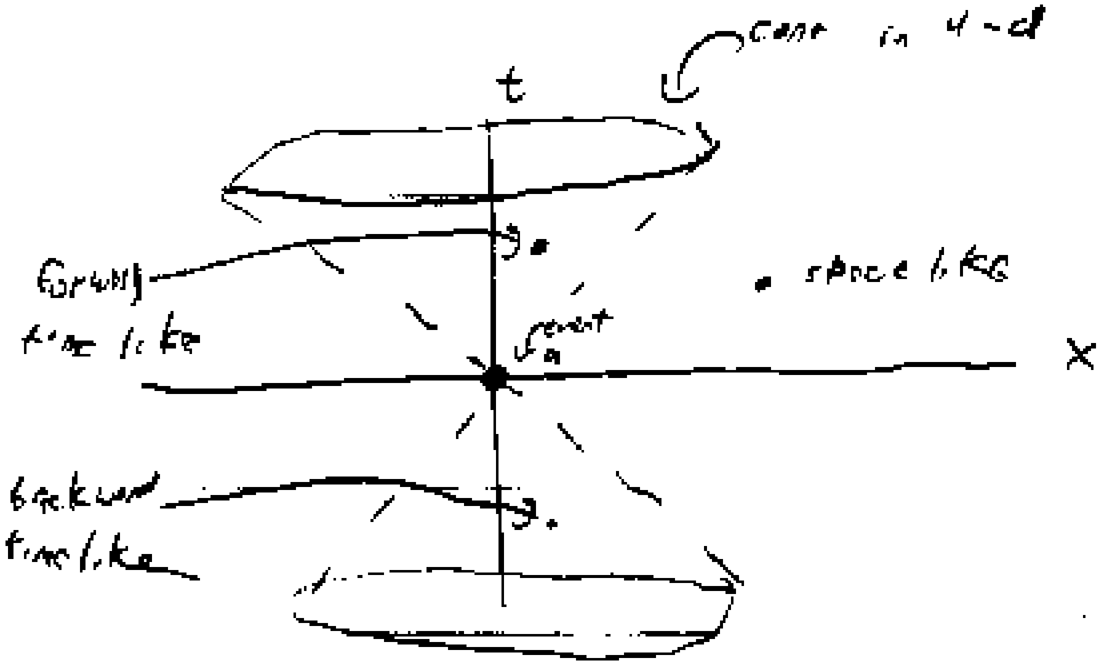
i) $\tau^2 < 0$ spacelike
ii) $\tau^2 > 0$ timelike
iii) $\tau^2 = 0$ light like

- If two events are spacelike separated event a and event b are not causally connected b could not have caused a or the other way around

- if two events are time like connected they can be causally connected

why? if two events are space like separated news of the 1st event can't get to second even traveling at speed of light proof i.e. $T^2 < 0$, $\Delta t^2 < \Delta \vec{x}^2$ but ~~top~~ information traveling at speed of light only covers a distance Δt $(c=1)$ so event 2 is farther away than light could have traveled

— think about Battle of New Orleans



why cone add y z get 4-d cone
light cone separates causal and acausal regions

Actually if $\Delta\tau^2 < 0$ (spacelike) it is ambiguous which event came first (frame dependent)

relativity of simultaneity

suppose two events seperated by L happen at same time in some frame

$$a = \begin{pmatrix} t=0 \\ x=0 \\ y=0 \\ z=0 \end{pmatrix} \qquad b = \begin{pmatrix} t=0 \\ x=L \\ y=0 \\ z=0 \end{pmatrix}$$

$$\Delta x^\mu = \begin{pmatrix} 0 \\ L \\ 0 \\ 0 \end{pmatrix}$$

boost

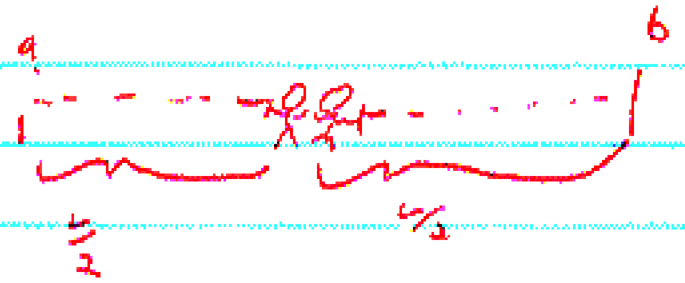
$$\Delta x'^\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ L \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\beta\gamma L \\ \gamma L \\ 0 \\ 0 \end{pmatrix}$$

in primed frame

$\Delta t' = -\beta\gamma L \neq 0$ events don't happen at same time - b happened first (positive why physically?

Western movie shootout — with lasers

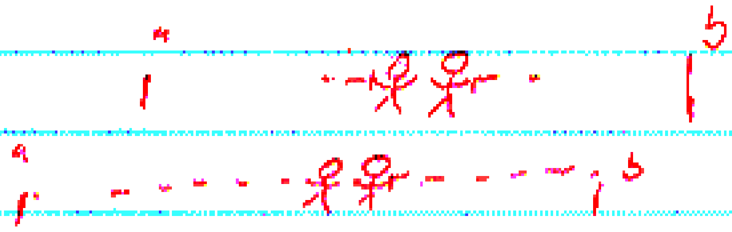
in frame of train



lasers shot at same time

hit walls a and b at same time $t = L/2$ (units of

but in moving frame



shoot
but train moves
before light hits wall

hits b before a

thus if it simultaneous in one frame not
in another

- spacelike — can find a frame where they occur at same time, with $t_a > t_b$ or $t_b > t_a$

- time like — can find frame where $x_a = x_b$

properties of Lorentz trans.

- ~~order~~ consecutive trans leads to matrix product

$$\vec{X}_1^{\,\mu} = \overleftrightarrow{\Lambda}_1 \vec{X}_0^{\,\mu}$$

$$\vec{X}_2^{\,\mu} = \overleftrightarrow{\Lambda}_2 \vec{X}_1^{\,\mu}$$

$$\vec{X}_2^{\,\mu} = \left(\overleftrightarrow{\Lambda}_2 \overleftrightarrow{\Lambda}_1\right) X_1^{\,\mu}$$

general Lorentz transform —

- rotation (keeps t same and $x^2+y^2+z^2$ same)
- boost in any direction (formally rotate to x boost in x rotate back

- We can use this fact to derive velocity addition formula (homework)

$$V_f = \frac{V_1 + V_2}{1 + V_1 V_2} \quad \text{if } V_1, V_2 \text{ are in same direc}$$

- cute fact

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ can be written in following way

$\gamma = \cosh \xi$

$\beta\gamma = \sinh \xi$ where $\xi \equiv$ rapidity

proof

$$\gamma^2 - (\beta\gamma)^2 = \gamma^2(1-\beta^2) = \frac{1}{1-\beta^2}(1-\beta^2) = 1$$

by definition

if we make identification above we

$$\gamma^2 - (\beta\gamma)^2 = \cosh^2\xi - \sinh^2\xi = 1$$ by general property of hyperbolic

Fact is not suprising

- Lorentz interval $\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta^2$ differs from a Euclidean 4-D metric by a minus sign only

- Lorentz transformation differs from rotation only by a couple of minus signs

Lorentz boost $\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$ with $\gamma^2 - \beta\gamma^2 = 1$

rotation $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$

but a rotation is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

so it's not shocking that boosts can be written in terms of hyperbolics

Cute fact - 2 while velocities do not add rapidities do

$$\beta\gamma = \sinh(\xi)$$
$$\gamma = \cosh(\xi)$$

$$\beta = \tanh(\xi) \quad \text{or} \quad \xi = \tanh^{-1}(\beta)$$

Now for two subsequent boosts in same direction

$$\beta_T = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}$$

so

$$\xi_T = \tanh^{-1}\left(\frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}\right)$$

$$= \tanh^{-1}\left(\frac{\tanh(\xi_1) + \tanh(\xi_2)}{1 + \tanh(\xi_1)\tanh(\xi_2)}\right)$$

~~Now let us prove a simple identity:~~

$$\cancel{\tanh(\xi_1 + \xi_2) = \frac{e^{\xi_1+\xi_2} - e^{-(\xi_1+\xi_2)}}{e^{\xi_1+\xi_2} + e^{-(\xi_1+\xi_2)}}}$$

but tanh addition formula

$$\tanh(\xi_1+\xi_2) = \frac{\tanh(\xi_1)+\tanh(\xi_2)}{1+\tanh(\xi_1)\tanh(\xi_2)}$$ (prove from def of tanh]

so

$$\xi_T = \tanh^{-1}\left(\tanh(\xi_1+\xi_2)\right) = \xi_1+\xi_2 \qquad Q.E.D$$

$\therefore$ radipities add

Other properties of Lorentz transformation

if

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

then

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & +\beta\gamma & 0 & 0 \\ +\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}$$

proof:

Physically - boosting by $+v$ is undone by boosting by $-v$

Mathematically

$$\begin{pmatrix} \gamma & +\beta\gamma & 0 & 0 \\ +\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2 - \beta^2\gamma^2 & -\beta\gamma^2 + \beta\gamma^2 & 0 & 0 \\ +\beta\gamma^2 - \beta\gamma^2 & -\beta^2\gamma^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

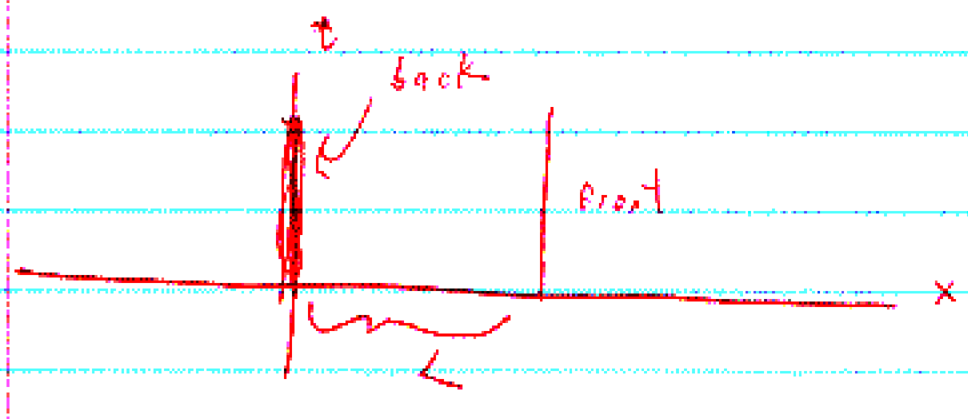
Matrices for a left boost and a right boost are inverses of each other!

use property of Lorentz transform to derive Lorentz contraction

in rest frame

$x_{front} = L$ for any time

$x_{back} = 0$



consider "event" of front of rod being at L at time t in rest frame in moving frame

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ L \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma t - \beta\gamma L \\ -\beta\gamma t + \gamma L \\ 0 \\ 0 \end{pmatrix}$$

I wish to find $x'_{front}(t')$ i.e. value of x for a known t' in moving frame

$x'_{front} = -\beta\gamma t + \gamma L$ but t is not in primed frame to get unprimed in terms of prime use inv

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & +\beta\gamma & 0 & 0 \\ +\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ 0 \\ 0 \end{pmatrix}$$

$$t = \gamma t' + \beta\gamma x'$$

so

$$x'_{front} = -\beta\gamma\left(\gamma t' + \beta\gamma x'\right) + \gamma L$$

$$(1 + \beta^2\gamma^2)\, x'_{front} = \gamma L - \beta\gamma^2 t'$$

$$x'_{front} = \frac{\gamma L - \beta\gamma^2 t'}{1 + \beta^2\gamma^2} = \frac{\gamma}{1+\beta^2\gamma^2} L - \frac{\beta t'}{\frac{1}{\gamma^2} + \beta^2} = \frac{\gamma L}{1+\beta^2\gamma^2} - \beta t'$$

do the same for back

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma t \\ -\beta\gamma t \\ 0 \\ 0 \end{pmatrix}$$

$$x'_{back} = -\beta\gamma t = -\beta t'$$

$$t'_{back} = \gamma t$$

In primed frame front and back both moving velocity of $-\beta$ (duh!)

$L' = x'_{front} - x'_{back}$ at same time in the frame (t')

$= \frac{\gamma L}{1+\beta^2\gamma^2} \cancel{\beta t} - (-\cancel{\beta t})$

$= \frac{\gamma L}{1+\frac{\beta^2}{1-\beta^2}} = \frac{\gamma L}{\frac{1-\beta^2}{1-\beta^2}+\frac{\beta^2}{1-\beta^2}} = \frac{\gamma L}{\frac{1}{1-\beta^2}} = (1-\beta^2)\gamma L = \gamma^{-2}\gamma L$

$= L/\gamma$

Note relativity of simultaneity plays key role

~~What is a vector~~

Similarly time dilation

suppose in rest frame we have ticking clock event are ticks

$$\begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

in moving frame we have

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma t \\ -\beta\gamma t \\ 0 \\ 0 \end{pmatrix}$$

so $t' = \gamma t$

thus moving observer says the clock runs slow — it take γt secs for the clock to make t ticks

consider two frames moving appart at velocity V. each say the other's clock runs slow

who is right?

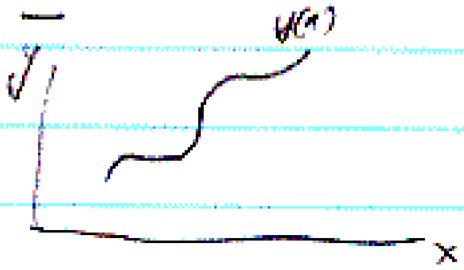
both!!

twin paradox — use concept of proper time to resolve

Key – study trajectories in space-time

First consider a curve in ordinary 3-d Euclidean space —

one way to parametrize —



$y = y(x)$
$z = z(x)$

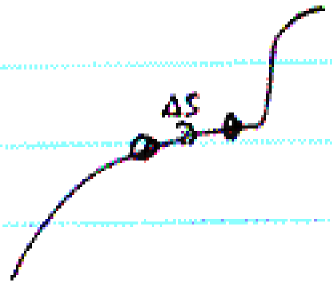
as you move along x axis position in y, z move

• such a parameterization sucks as far as transformation properties of space under rotations is concerned: x, y, z are created equal as far as space is concerned

• a better parametization — parameterize curve in terms of a single scalar — useful choice is s, the arc length along the curve

$x(s)$
$y(s)$
$z(s)$

Now the relation between position and arclength is essentially different.



$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \qquad \text{Pythagoras}$$

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

so if I have things written in terms of x, $y(x)$, $z(x)$

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx$$

can now express Δ arclength in terms of dx

or

$$s = \int_{x_0}^{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx$$

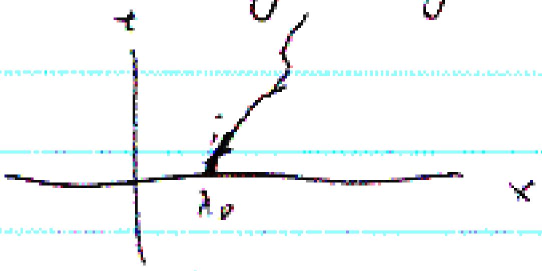
so s is length along curve from start point — it is a scalar
rotate curve but s stays same —
length doesn't depend on orientation

Now consider a trajectory in space-time

- motion of object — each point along this path is an "event"

non relativistically we generally write

$x = x(t)$
$y = y(t)$
$z = z(t)$



analog of $y(x)$, $z(x)$ for a 3-d curve —

sucks — difficult to consider transformations of space time since t, x mix

trick — parameterize curve by a Lorentz invariant measure (same in all frames) also called a Lorentz scalar

this scalar is the proper time along path, τ

~~recall for any two events~~

$\cancel{\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2}$

this path in space time parameterized by

$t(\tau)$
$x(\tau)$
$y(\tau)$
$z(\tau)$

how can I find τ along path.
recall for any two events

$$\Delta\tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

consider infinitesimally close events

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$

so

$$d\tau = \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$

$$= \sqrt{1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}\ dt$$

$$= \sqrt{1 - v^2}\quad dt$$

$$d\tau = \frac{dt}{\gamma(t)} \qquad \text{or} \qquad dt = \gamma\ \tau$$

where $\gamma(t) = \frac{1}{\sqrt{1-v^2(t)}}$ is γ factor for a frame comoving with object relative to lab frame

issues — possible problem

$d\tau = \sqrt{1-\vec{v}^2}\, dt$

- could be complex
- for a general curve in space-time this could happen

. but for motion of physical particles

$\vec{v} < 1$ $\sqrt{1-\vec{v}^2}$ is always real

— τ for any point along trajectory is the same in any frame by construction

— interpretation of τ

- at any given instant there exists an inertial frame comoving with object
  - in this comoving frame object is at rest so $dt_{comoving} = d\tau$
- clock carried with object will tick at same rate as a clock in a comoving frame

$\therefore$ τ is the time elapsed on a clock moving with object twin paradox

trajectory of twin 1 in "Earth frame"

$t = \tau_1$
$x = 0$
$y = 0$
$z = 0$

trajectory of twin 2 in "earth frame" on way out (till $\tau_{max}/2$)

$t = t$
$x = +vt$
$y = 0$
$z = 0$

on way back

$t = t$
$x = x_{max} - vt$
$y = 0$
$z = 0$

let us parametrize this 2^{nd} twins path by τ_2

on way out

$d\tau = \sqrt{1-v^2}\, dt$
$t = \gamma \tau_2$
$x = v\gamma \tau_2$
$y = 0$
$z = 0$

on way back

$t = \gamma \tau_2$
$x = x_{max} - v\gamma(\tau_2 - \tau_{max})$ where $x_{max} = v\gamma\tau_{max}$
$y = 0$
$z = 0$

get home at time of $\tau = 2\tau_{max}$
$x = 0$

but in earth frame

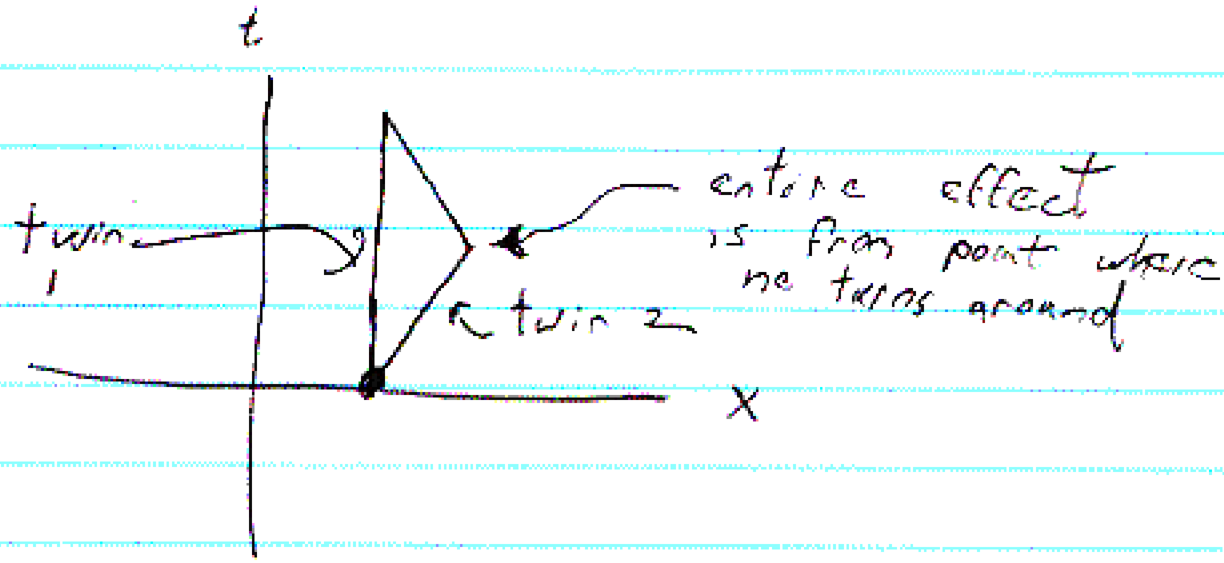
$t = 2\gamma\tau_{max}$

in Rocket's clock $2\tau_{max}$ has elapsed
on Earth $2\tau_{max}\gamma$
the moving twin is younger

~ Fountain of Youth?

~ paradox why don't we do it from frame of moving Astronaut – then he is still and his brother younger

but...
he is not in any single inertial frame for problem



if we had computed from frame comoving with twin 2 on the way out the proper times for the two trajectories of $2\tau^0_{max}$ and $(2T_{2max})\gamma$ better be the same. (Homework)

Concept of a 4-vector (Lorentz Vector) and a Lorentz Scalar [Lorentz invariant]

- ordinary Euclidean vectors and scalars
  - vector is a set of 3 numbers which transforms as a vector under rotations
  - scalar is a number which is invariant under rotations

not any random collection of 3 numbers
is not a vector

eg define the following "vector" to
~~describe~~ describe a person

$$\vec{Person} = \begin{pmatrix} \text{age} \\ \text{shoe size} \\ \text{GPA} \end{pmatrix}$$

clearly not a vector

eg define the following "scalar"

$$s = x\text{-component of position}$$

a real vector transforms the same way as $\vec{x}$ under ...

$$\vec{x}' = \overleftrightarrow{R}\,\vec{x}$$

$$\vec{v}' = \overleftrightarrow{R}\,\vec{v}$$

rotation matrix
3x3 orthogonal matrix

$s' = s$ (invariant under rotation)

e.g. $\vec{x}, \vec{p}, \ldots$

as $s = \sqrt{(\vec{x}_2 - \vec{x}_1)\cdot(\vec{x}_2 - \vec{x}_1)}$

to get a scalar from two vectors take the dot product $s = \vec{V}_1 \cdot \vec{V}_2$

by analogy —

- a Lorentz scalar is a quantity which is the same in all inertial frames

- a Lorentz vector is a set of 4 numbers which transforms under Lorentz transformations in same way as $\vec{x}^4$

- so by definition $\vec{x}^4$ is a 4-vector!
- τ by construction is a Lorentz scalar

to define more we need a Lorentz dot product

claim: consider two 4-vectors

$$\left.\begin{aligned} \vec{A}'^4 &= \overset{\text{Lorentz tran}}{\Lambda}\,\vec{A}^4 \\ B'^4 &= \Lambda\, B^4 \end{aligned}\right\} \text{ under Lorentz transformation (def. of 4 vectors)}$$

when

$$\vec{A}^4 = \begin{pmatrix} A_0 \\ A_x \\ A_y \\ A_z \end{pmatrix} \qquad \vec{B}^4 = \begin{pmatrix} B_0 \\ B_x \\ B_y \\ B_z \end{pmatrix}$$

then
$$A_0 B_0 - A_x B_x - A_y B_y - A_z B_z$$
$$= A_0' B_0' - A_x' B_x' - A_y' B_y' - A_z' B_z'$$

proof — identical to proof in homework for case of $t^2-x^2-y^2-z^2$ (same algebra)

that combo is an invariant or scalar just as dot product is in Euclidean space

Lorentz dot product:

$$\vec{A}^4 \cdot \vec{B}^4 \equiv A_0B_0 - A_xB_x - A_yB_y - A_zB_z$$

$$(\Delta\tau)^2 = \vec{\Delta X}^4 \cdot \vec{\Delta X}^4$$

makes a four vector —

How can we exploit ideas of 4 vectors, Lorentz scalars?

eg. consider a plane wave traveling at speed of light

$$\psi(x,t) = A e^{i\, \vec{k}\cdot\vec{x} - \omega t}$$

where $\frac{\omega}{k} = c = 1$

claim the phase of the wave $\phi = i\vec{k}\cdot x - \omega t$ must be a Lorentz scalar

why? event in space time where wave achieves min or max or whatever is same in all frames

so $\phi = \omega t - kx$

ϕ is a Lorentz scalar

but this means

$$\vec{k}^4 \equiv \begin{pmatrix} \omega \\ k_x \\ k_y \\ k_z \end{pmatrix}$$ must be a 4 vector

since

$$\phi \equiv \vec{k}^4 \cdot \vec{x}^4 = \omega t - kx$$

and only 4 vector doted with 4 vector gives a Lorentz scalar

now the fact that

$\vec{k}^4$ is a 4 vector allows us to deduce what happens to $\omega, \vec{k}$ as one shifts frame assume light wave propagating in x direction

$$\vec{k} = k\hat{x} = \omega\hat{x} \qquad \text{as } \omega = k$$

$$\vec{k}^4 = \begin{pmatrix} \omega \\ \omega \\ 0 \\ 0 \end{pmatrix}$$

— cute fact $\vec{k}^4 \cdot \vec{k}^4 = 0$ true for all light waves

— boost system in x direction (same direction as propagation)

$$\vec{k}^{4\prime} = \begin{pmatrix} \omega' \\ k_x' \\ k_y' \\ k_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega \\ \omega \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(1-\beta)\omega \\ \gamma(1-\beta)\omega \\ 0 \\ 0 \end{pmatrix}$$

not $k^{4\prime} \cdot k^{4\prime} = 0$ same as in unprimed frame

now what is ω'? $\omega' \neq \omega$ Doppler shift

$$\omega' = \gamma(1-\beta)\omega \qquad \text{frequency is shifted}$$

$$= \frac{1-\beta}{\sqrt{1-\beta^2}}\,\omega = \frac{1-\beta}{\sqrt{(1-\beta)(1+\beta)}}\,\omega = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega$$

So $\omega' < \omega$ red shift

red shift essential in Astro to measure speed of stars

– run away from light source → lowers frequency

what happens as $\beta \to \infty$

$\omega \to 0$

what if we boost in opposite direction (we run toward light source)

same thing but $\beta \to -\beta$

$\omega' = \sqrt{\frac{1+\beta}{1-\beta}}\ \omega$ $\quad \omega' > \omega$ blue shift

Note for $\beta \ll 1$

$\omega' = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega \approx (1-\beta)\ \omega$

~~Doppler shift from vectors~~

Note that no matter how fast we run we can't go to a frame which makes a left propagating wave right propagating $(-\omega)$

– transverse Doppler shift? (Homework)

– Doppler shift for wave traveling ~~space~~ with phase velocity $v \neq c$
eg sound, quantum wave for pion …

$\phi = \vec{\omega t} - \vec{k}\vec{x} = \vec{k}^4 \cdot \vec{x}^4$

Lorentz scalar

4 vector

thus $\vec{k}^4$ is still a 4 vector

Difference $\frac{\omega}{k} = v$ so $k = \omega/v$

$$\vec{k}^4 = \begin{pmatrix} \omega \\ \omega/v \\ 0 \\ 0 \end{pmatrix}$$

– $\vec{k}^4 \cdot \vec{k}^4 = \omega^2(1 - 1/v^2)$
same in all frames

suppose we boost (eg sound wave)

$$\begin{pmatrix} \omega' \\ k_x' \\ k_y' \\ k_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega \\ \omega/v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(1 - \beta/v)\omega \\ \gamma(-\beta + 1/v)\omega \\ 0 \\ 0 \end{pmatrix}$$

explicitly find

$\vec{k}^{4\prime} \cdot \vec{k}^{4\prime} = \gamma^2(1 - \beta/v)^2\omega^2 - \gamma^2(-\beta + \frac{1}{v})^2\omega^2$

$$\vec{k}^{4\prime}\cdot\vec{k}^{4\prime} = w^2 \frac{1}{(1-\beta^2)}\left[\left(1 - \frac{2\beta}{v} + \frac{\beta^2}{v^2}\right) - \left(\beta^2 - \frac{2\beta}{v} + \frac{1}{v^2}\right)\right]$$

$$= \frac{w^2}{(1-\beta^2)}\left[(1-\beta^2) - \frac{1}{v^2}(1-\beta^2)\right] = w^2\left(1 - \frac{1}{v^2}\right)$$

same as $\vec{k}^4\cdot\vec{k}^4$!!

now Doppler shift

$$w' = \gamma\left(1 \underset{\text{blue}}{\overset{\text{red}}{\mp}} \beta/v\right) w$$

note that for $\beta > v$

w' is negative — wave propagates backwards in my frame

but for light this is not possible

— other 4 vectors

consider a trajectory of a body in Minkowski space — parametrize path by proper time (which is an invariant)

$$\vec{x}^4(\tau) = \begin{pmatrix} t(\tau) \\ x(\tau) \\ y(\tau) \\ z(\tau) \end{pmatrix}$$

construct

$$\vec{u}^4 \equiv \frac{d\vec{x}^4}{d\tau} \qquad \begin{pmatrix} u_0 \\ u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} dt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix}$$

claim u is a four vector — derivative of a vector wrt a scalar is a vector

i.e.

$$\begin{pmatrix} u_0' \\ u_x' \\ u_y' \\ u_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \gamma u_0 - \beta\gamma u_x \\ -\beta\gamma u_0 + \gamma u_x \\ u_y \\ u_z \end{pmatrix}$$

properties of u

- $\vec{u}^4 \cdot \vec{u}^4 = 1$

Proof: go to a frame in which at a given instant object is at rest

then four velocity is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ in that frame

as $\frac{dt}{d\tau} = 1$

thus in that frame $\vec{u}^4 \cdot \vec{u}^4 = 1$

but it is a scalar so its true in all frames

- a^μ can be written as

$$u^\mu = \begin{pmatrix} \gamma \\ v_x\gamma \\ v_y\gamma \\ v_z\gamma \end{pmatrix} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-v^2}}$$

satisfies

$$\vec{u}^\mu \cdot \vec{u}^\mu = 1$$

proof

$$\vec{u}^\mu \cdot \vec{u}^\mu = \gamma^2 - v^2\gamma^2 = \frac{1-v^2}{1-v^2} = 1 \quad \text{Q.E.D}$$

- for slowly moving objects in some frame

$$\gamma = 1 + O(v^2)$$
$$\vec{v}\gamma = \vec{v} + O(v^2)$$

$$u^\mu = \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix} + O(v^2)$$

- rest frame $u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

— Note that $\vec{u}^4 \cdot \vec{u}^4$ is same in all frames since 1 is invariant!

— We can use $\vec{u}^4$ to derive velocity addition formula

$$\vec{u}^4 = \begin{pmatrix}\gamma\\ \underbrace{\gamma\vec{v}}_{\text{3 space components}}\end{pmatrix} = \begin{pmatrix}\gamma\\ \gamma v_x\\ \gamma v_y\\ \gamma v_z\end{pmatrix}$$

$$v_i = \frac{u^4_i}{u^4_0}$$

— consider a particle moving with velocity v in the "1" frame (assume v is in $+\hat{x}$ direction) go to a frame "2" moving $-\hat{x}$ direction by an amount V (so velocities add)

$$\vec{u}^4_1 = \begin{pmatrix}\gamma_1\\ \gamma_1 v\\ 0\\ 0\end{pmatrix}$$

$$\vec{u}^4_2 = \begin{pmatrix}\gamma_2\\ \gamma_2 v_{x2}\\ \gamma_2 v_{y2}\\ \gamma_2 v_{z2}\end{pmatrix} = \begin{pmatrix}\gamma & +V\gamma & 0 & 0\\ +V\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_1 v_1\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}\gamma\gamma_1(1+Vv_1)\\ \gamma\gamma_1(V+v_1)\\ 0\\ 0\end{pmatrix}$$

so $v_{2x} = \dfrac{u^4_x}{u^4_0} = \dfrac{(V+v_1)\,\gamma_1\gamma_2}{(1+Vv_1)\,\gamma_1\gamma_2} = \dfrac{V+v_1}{1+Vv_1}$ ✓

Covariant acceleration

$$\vec{a}^4 \equiv \frac{d\vec{u}^4}{d\tau}$$

- properties

— $\vec{a}^4 \cdot \vec{u}^4 = 0$

proof: $\frac{d}{d\tau}\vec{u}^4 \cdot \vec{u}^4 = \frac{d}{d\tau} 1 = 0$

$= 2\frac{d\vec{u}^4}{d\tau} \cdot \vec{u}^4 = 2\,\vec{a}^4 \cdot \vec{u}^4$ Q.E.D.

— in an instantaneous rest frame ($\vec{u}^4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$) fact that $\vec{a}^4 \cdot \vec{u}^4 = 0$ implies that $\vec{a}^4 = \begin{pmatrix} 0 \\ \vec{a} \end{pmatrix}$

where that $\vec{a}$ is $\frac{d\vec{v}}{dt}$ (the usual acceleration) why ~~[illegible]~~ for velocities that are small

$\vec{u}^4 = \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} + O(\vec{v}^2)$

$t = \tau + O(v^2)$

so $\frac{\partial u^4}{d\tau} = \frac{du^4}{dt} + O(v^2) = \begin{pmatrix} 0 \\ \frac{d\vec{v}}{dt} \end{pmatrix} + O(v^2)$ Q.E.D.

So in comoving frame of object

$$\vec{a}^4 = \begin{pmatrix} 1 \\ \vec{a} \end{pmatrix}$$ i.e. this gives acceleration experienced by object

$\therefore$ $\vec{a}^4 \cdot \vec{a}^4 = -a^2$ where a is acceleration in comoving frame

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eg Consider a rocket starting from rest in some frame which accelerates in $\hat{x}$ directions. The acceleration feels constant (independent of time) to the people on rocket. Find the rocket's velocity as a function of time

$$\frac{d\vec{u}^4}{d\tau} = \vec{a}^4$$

boost into frame moving with Rocket

a is rest frame of rocket

but
$$\vec{a}^4 = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{d}{d\tau}\begin{pmatrix} \frac{1}{\sqrt{1-v^2}} \\ \frac{v}{\sqrt{1-v^2}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{va}{\sqrt{1-v^2}} \\ \frac{a}{\sqrt{1-v^2}} \\ 0 \\ 0 \end{pmatrix}$$

trick

$$\frac{1}{\sqrt{1-v^2}} \equiv \cosh(\eta)$$
$$\frac{v}{\sqrt{1-v^2}} \equiv \sinh(\eta)$$

use rapidity

$$\frac{d}{d\tau}\begin{pmatrix}\cosh(\eta)\\ \sinh(\eta)\\ 0\\ 0\end{pmatrix} = a\begin{pmatrix}\sinh(\eta)\\ \cosh(\eta)\\ 0\\ 0\end{pmatrix}$$

or

$$\begin{pmatrix}\sinh(\eta)\,\frac{\partial \eta}{\partial \tau}\\ \cosh(\eta)\,\frac{\partial \eta}{\partial \tau}\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}a\,\sinh(\eta)\\ a\,\cosh(\eta)\\ 0\\ 0\end{pmatrix}$$

$$\therefore \quad \frac{d\eta}{d\tau} = a$$

$\eta = a\tau$ rapidity grows linearly with time

~~fun~~ ~~ccd~~ $\vec{u}^{\nu} = \begin{pmatrix}\cosh(a\tau)\\ \sinh(a\tau)\\ 0\\ 0\end{pmatrix}$

but I want

$v(t)$ in original

$$v = \frac{u^1}{u^0} = \tanh(a\tau)$$

$$dt = \gamma(\tau)\,d\tau = \frac{1}{\sqrt{1-v^2}}\,d\tau$$

$$t = \int \cosh(a\tau)\,d\tau$$

$$t = \frac{1}{a}\sinh(a\tau)$$

or

$$at = \sinh(a\tau)$$

$$\tau = \frac{1}{a}\sinh^{-1}(a\tau)$$

$$v = \tanh(a\tau) = \tanh(\sinh^{-1}(at))$$

$$= \frac{\sinh(\sinh^{-1}(at))}{\cosh(\sinh^{-1}(at))} = \frac{at}{\sqrt{1+\sinh^2(\sinh^{-1}(at))}} = \frac{at}{\sqrt{1+(at)^2}}$$

limits : for small times $at << 1$

$v = at + \mathcal{O}(t^3)$ Non-relativistic limit

long times $at >> 1$

~~[illegible]~~

$$v = \frac{1}{\sqrt{1+\left(\frac{1}{at}\right)^2}} = 1 - \frac{1}{2a^2t^2}$$

$v \to 1$ why

Now suppose we multiply $\vec{u}^4$ by the mass of the object

which mass? the rest mass or "kinetic mass"
The mass (in some really bad books called rest mass)

the mass is a property of object and by definition is a Lorentz scalar

let me call this new 4-vector

$$\vec{p}^4 = m\vec{u}^4$$

but what is this 4-velocity

$$\vec{p}^4 = \begin{pmatrix} m\gamma \\ m\vec{v}\gamma \end{pmatrix}$$

to understand interpretation look at low velocity limit in some frame up to order v^2

$$\gamma = \frac{1}{\sqrt{1-v^2}} = 1 + \frac{1}{2}v^2 + \mathcal{O}(v^4)$$

$$\vec{v}\gamma = \vec{v} + \mathcal{O}(\vec{v}^3)$$

$$\vec{p}^{\,4} = \begin{pmatrix} m(1+\frac{1}{2}v^2) \\ mv \end{pmatrix} + \mathcal{O}(v^3)$$

$$= \begin{pmatrix} m + \frac{1}{2}mv^2 \\ m\vec{v} \end{pmatrix}$$

interpretation — lower three components are $\vec{p}$ non relativistic

$\therefore$ $\vec{p}^{\,4}_x, \vec{p}^{\,4}_y, \vec{p}^{\,4}_z \equiv \vec{p}$ so $\boxed{\vec{p} \equiv m\vec{v}\gamma}$

0^{th} component $= m + \frac{1}{2}mv^2$

$= m +$ Kinetic Energy

natural to identify 0^{th} component with Energy as it agrees in non-relativistic limit

$E = m\gamma$

but...

at $v=0$ $\gamma=0$ and then $E=m$

Putting back units

$E = mc^2$

but note this is not a general result. Only good for objects at rest

general

$\vec{p}^4 \cdot \vec{p}^4 = m^2$

$$\boxed{E^2 - \vec{p}^{\,2} = m^2}$$

or

$$E = \sqrt{m^2 + p^2}$$

This plus conservation of energy allows us to work out results of collisions and decays

example —

a particles of mass, m and ~~mass~~ velocity v decays into two particles — one is measured — How do we find mass of second initial 4 momentum

$$\begin{pmatrix} E \\ \vec{p} \end{pmatrix} = \begin{pmatrix} E_1 \\ \vec{p}_1 \end{pmatrix} + \begin{pmatrix} E_2 \\ p_2 \end{pmatrix}$$

$$\vec{p}^{\,4} = \vec{p}_1^{\,4} + \vec{p}_2^{\,4}$$

so

$$\vec{p}_2^{\,4} = \vec{p}^{\,4} - \vec{p}_1^{\,4}$$

so

$$m_2^2 = \vec{p}_2^{\,4} \cdot \vec{p}_2^{\,4} = \left(\vec{p}^{\,4} - \vec{p}_1^{\,4}\right) \cdot \left(p^4 - \vec{p}_1^{\,4}\right)$$

$$= p^4 \cdot p^4 + \vec{p}_1^{\,4} \cdot \vec{p}_1^{\,4} - 2\,\vec{p}^{\,4} \cdot \vec{p}_1^{\,4}$$

$$= m^2 + m_1^2 - 2\,\vec{p}^{\,4} \cdot \vec{p}_1^{\,4}$$

$$m_2 = \sqrt{m^2 + m_1^2 - 2\left(E E_1 - p \cdot \vec{p}_1\right)}$$

$\therefore$ by knowing original $\vec{v}$ and $\hat{M}$
and knowing $\vec{E}_1$ and $\vec{p}_1$ we know $\vec{E}$

let's look at decay in a particular frame

suppose a particle of M at rest breaks into 2 equal mass (m) particle. How fast do they go and what is their energy and momentum

$$\vec{P}^{4} = \vec{P}_1^{4} + \vec{P}_2^{4}$$

$$\begin{pmatrix} M \\ \vec{0} \end{pmatrix} = \begin{pmatrix} \sqrt{m^2+p_1^2} \\ \vec{p}_1 \end{pmatrix} + \begin{pmatrix} \sqrt{m^2+p_2^2} \\ \vec{p}_2 \end{pmatrix}$$

but $\vec{P}_1 = -\vec{P}_2 \equiv \vec{P}_d$ decay

$$M = 2\sqrt{m^2+p_d^2}$$ $\therefore$ each has energy $\frac{M}{2}$

$$\left(\frac{M}{2}\right)^2 = m^2 + P_d^2$$

$$\hat{P}_d = \sqrt{\left(\frac{M}{2}\right)^2 - m^2}$$

$$V_d = \frac{P_d}{E} = \frac{m v \gamma}{m \gamma} = \frac{\sqrt{\left(\frac{M}{2}\right)^2 - m^2}}{\frac{M}{2}}$$

Two special cases

$m \to 0$

$v_d \to 1$

massless particles alway move at $v = 1$ (c)

- $m = \frac{M}{2} - \Delta$ with $\Delta << M$

$$\sqrt{\left(\frac{M}{2}\right)^2 - m^2} = \sqrt{\left(\frac{M}{2}\right)^2 - \left(\frac{M}{2} - \Delta\right)^2}$$

$$= \sqrt{\cancel{\left(\frac{M}{2}\right)^2} - \left(\cancel{\left(\frac{M}{2}\right)^2} - 2\Delta\left(\frac{M}{2}\right) + \Delta^2\right)} = \sqrt{M\Delta + \Delta^2}$$

$$= \sqrt{M\Delta}\ \sqrt{1 + \frac{\Delta^2}{M\Delta}} = \sqrt{M\Delta}\ \sqrt{1 + \frac{\Delta}{M}}$$

$$= \sqrt{M\Delta}\ \left(1 + O\left(\frac{\Delta}{M}\right)\right)$$

so

$$v_d = \frac{\sqrt{\left(\frac{M}{2} - m\right)^2}}{\frac{M}{2}} \approx \frac{2}{M}\sqrt{M\Delta}$$

$$= 2\sqrt{\frac{\Delta}{M}}$$

Note this usual nonrelativistic result

The kin. energy given to each particle is Δ and ~~their~~ their mass is $M/2$ (in this limit

so

$\frac{1}{2} m V_d^2 = KE$

$$V_d = \sqrt{\frac{2KE}{m}} = \sqrt{\frac{2\Delta}{\frac{M}{2}}} = 2\sqrt{\frac{\Delta}{M}}$$

---

decay in rest frame of M gave each particle moving at v_d

what about moving M ?

Assume M has velocity $\vec{v}$ in lab frame

— issues • what direction are the decay particles moving? How fast?

(in rest frame they could go in any direction back-to-back

Easy way to do this —

two steps — • work in rest frame of M (c.m. frame)
• boost result to frame of M moving with v

in rest frame of M $\quad v_d = \dfrac{\sqrt{\left(\frac{M}{2}\right)^2 - m^2}}{\frac{M}{2}}$

and the two particles have momenta

$$P_d = \sqrt{\left(\frac{M}{2}\right)^2 - m^2}$$

and Energy $\frac{M}{2}$

suppose the go back to back in x-y plane

$$\vec{p}^{\,4}_{1_{c.m.}} = \begin{pmatrix} \frac{M}{2} \\ P_d\cos\theta \\ P_d\sin\theta \\ 0 \end{pmatrix} \qquad \vec{p}^{\,4}_{2_{c.m.}} = \begin{pmatrix} \frac{M}{2} \\ -P_d\cos\theta \\ -P_d\sin\theta \\ 0 \end{pmatrix}$$

in a frame moving with velocity $-V$ in x-direction (boosting M)

$$\vec{P}^{\,4}_{1\,lab} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{M}{2} \\ P_d\cos\theta \\ P_d\sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\gamma M}{2} + \beta\gamma P_d\cos\theta \\ \frac{\beta\gamma M}{2} + \gamma P_d\cos\theta \\ P_d\sin\theta \\ 0 \end{pmatrix}$$

$$\vec{P}^{\,4}_{2\,lab} = \begin{pmatrix} \frac{\gamma M}{2} - \beta\gamma P_d\cos\theta \\ \frac{\beta\gamma M}{2} - \gamma P_d\cos\theta \\ P_d \sin\theta \\ \ldots \end{pmatrix}$$