PHYS 374 Homework #9

1. $a_m = \frac{1}{m}\int d^2x' \, \rho(x') \, r'^m e^{-i\phi m}$

when a_2 is a non-vanishing cofficient, $(a_0 = a_1 = 0)$

$$a_2 = \frac{1}{2}\int d^2x' \rho(x') r'^2 e^{i2\phi'}$$
$$= \frac{1}{2}\int d^2x' \rho(x') (r' e^{i\phi'})^2 = \frac{1}{2}\int d^2x' \rho(x') (r'\cos\phi' - i r'\sin\phi')^2$$
$$= \frac{1}{2}\int d^2x' \rho(x') (x' - iy')^2 = \frac{1}{2}\int d^2x' \rho(x') (x'^2 - 2ix'y' - y'^2)$$

Now, consider a shift of the origin,

$$x'' = x' - C_x \quad , \quad y'' = y' - C_y \quad \Rightarrow \quad dx'' = dx' \, , \; dy'' = dy'$$

Then. a_2 is

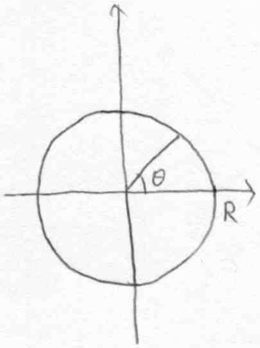
$$a_2 = \frac{1}{2}\int d^2x' \rho(x') \left[(x''+C_x)^2 - 2i(x''+C_x)(y''+C_y) - (y''+C_y)^2 \right]$$
$$= \frac{1}{2}\int dx'' dy'' \rho(x'') \left[x''^2 + 2x''C_x + C_x^2 - 2i(x''y'' + C_y x'' + C_x y'' + C_x C_y) - y''^2 - 2y''C_y - C_y^2 \right]$$
$$= \frac{1}{2}\int dx'' dy'' \rho(x'') \left[x''^2 - 2ix''y'' - y''^2 \right]$$
$$+ \frac{1}{2}\int dx'' dy'' \rho(x'') \left(2C_x x'' - 2iC_y x'' - 2iC_x y'' - 2C_y y'' \right) \longrightarrow a_1 = 0$$
$$+ \frac{1}{2}\int dx'' dy'' \rho(x'') \left(C_x^2 - 2iC_xC_y - C_y^2 \right) \rightarrow a_0 = 0$$
$$= \frac{1}{2}\int dx'' \int dy'' \rho(x'') \left[x''^2 - 2ix''y'' - y''^2 \right]$$
$$= a_2^{new}$$

$$\therefore \; a_2^{new} = a_2$$

$$\frac{1}{2}\int dx'' dy'' \rho(x'') (2C_x x'' - 2iC_y x'' - 2iC_x y'' - 2C_y y'') = \frac{1}{2}\underbrace{\int d^2x'' dy'' \rho(x'') (x'' - iy'')}_{a_1} (C_x - iC_y) \cdot 2$$
$$= a_1 \cdot (C_x - iC_y) = 0$$

$$\frac{1}{2}\underbrace{\int dx'' dy'' \rho(x'')}_{a_0} (C_x^2 - 2iC_yC_x - C_y^2) = a_0 \cdot \frac{1}{2}(C_x - iC_y)^2 = 0$$

2.



$$\Phi(R,\theta) = V_0 \left(\sin^2\theta - \frac{1}{2}\right)$$

a) $$\Phi(R,\theta) = V_0\left[\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 - \frac{1}{2}\right] = \frac{V_0}{4}\left[-e^{2i\theta} + \cancel{2} - e^{-2i\theta} - \cancel{2}\right]$$

$$= -V_0 \frac{1}{4}\left(e^{2i\theta} + e^{-2i\theta}\right)$$

Consider the general solution in polar coordinate,

$$\Phi(r,\theta) = b_0 \log\left(\frac{r}{r_0}\right) + \sum_{m=1}^{\infty} \frac{a_m}{r^m} e^{im\theta} + \sum_{m=1}^{\infty} b_m r^m e^{im\theta} + c.c.$$

Inside the cylinder, when $r \to 0$, $\frac{1}{r^m} \to \infty$. $\Rightarrow a_m = 0$ for all m

At $R = r$, $\Phi(R,\theta) = -\frac{V_0}{4}\left(e^{2i\theta} + e^{-2i\theta}\right)$

$\Rightarrow b_0 = 0$. $b_m = 0$ for $m \neq \pm 2$

$$b_m = \frac{1}{R^m}\frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-im\theta}\, \Phi(R,\theta)$$

$$= \frac{1}{R^m}\cdot\frac{1}{2\pi}\int_0^{2\pi} d\theta \left(-\frac{V_0}{4}\right)\left(e^{i(2\theta - m\theta)} + e^{-i(2+m)\theta}\right)$$

$$= \frac{1}{R^m}\cdot\frac{1}{\cancel{2\pi}}\left(-\frac{V_0}{4}\right)\cancel{2\pi}\left(\delta_{m,-2} + \delta_{m,2}\right)$$

$b_2 = -\frac{V_0}{4R^2}$, $b_n = 0$ for $m \neq \pm 2$.

$$\Phi(r,\theta) = \sum_m b_m r^m \left(e^{2i\theta} + e^{-2i\theta}\right)$$

$$= r^2\left(-\frac{V_0}{4R^2}\right)\cdot 2\cos 2\theta = -\frac{V_0}{2}\cdot\frac{r^2}{R^2}\cdot\cos 2\theta$$

b) $\vec{E} = -\nabla \Phi = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}\right)\Phi$.

$$= -\left[-\frac{V_0}{2}\frac{2r}{R^2}\cos 2\theta\, \hat{r} + \frac{1}{r}\left(+\frac{V_0}{2}\frac{r^2}{R^2}\right)(+2)\sin 2\theta\, \hat{\theta} \right]$$

$$= \frac{V_0 r}{R^2}\cos 2\theta\, \hat{r} - \frac{V_0 r}{R^2}\sin 2\theta\, \hat{\theta}$$

$$= \frac{V_0}{R^2} r\left(\cos 2\theta\, \hat{r} - \sin 2\theta\, \hat{\theta}\right)$$

c) for $r > R$,

when $r \to \infty$, $r^m \to \infty$. and $\frac{1}{r^m} \to 0$

In the general solution, $b_m = 0$ for all m.

$$a_m = \frac{1}{2\pi} R^m \int_0^{2\pi} d\theta\, e^{-im\theta}\, \Phi(R.\theta)$$

$$= \frac{1}{2\pi} R^m \int_0^{2\pi} d\theta\, e^{-im\theta}\left(-\frac{V_0}{4}\right)\left(e^{i2\theta} + e^{-i2\theta}\right)$$

$$= \frac{R^m}{2\pi}\frac{-V_0}{4}\int_0^{2\pi}\left(e^{i(2\theta - m\theta)} + e^{-i\theta(2+m)}\right) d\theta$$

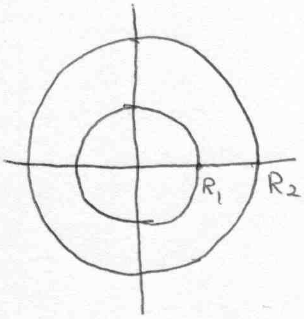
$$= -\frac{V_0 R^m}{8\pi}\cdot 2\pi\left(\delta_{m,2} + \delta_{m,-2}\right)$$

$$a_2 = -\frac{V_0}{4}R^2.$$

$$\Phi(r.\theta) = \sum_{m=0}^{\infty}\frac{a_m}{r^m}\left(e^{im\theta} + e^{-im\theta}\right)$$

$$= -\frac{V_0 R^2}{4 r^2}\, 2\cos 2\theta. = -\frac{V_0}{2}\frac{R^2}{r^2}\cos 2\theta = V_0\frac{R^2}{r^2}\left(\sin^2\theta - \frac{1}{2}\right)$$

3.



$$\Phi(R_1,\theta) = V_1\left(\sin^2\theta - \tfrac{1}{2}\right) = -\frac{V_1}{4}\left(e^{i2\theta} + e^{-i2\theta}\right)$$

$$\Phi(R_2,\theta) = V_2\left(\cos^2\theta - \tfrac{1}{2}\right) = +\frac{V_1}{4}\left(e^{i2\theta} + e^{-i2\theta}\right)$$

when $R_1 < r < R_2$,

The general solution is,

$$\Phi(r,\theta) = b_0 \log\frac{r}{r_0} + \sum_m \frac{a_m}{r^m} e^{im\theta} + \sum_m b_m r^m e^{im\theta} + \text{c.c.}$$

At $r = R_1$ or $r = R_2$, to satisfy the boundary condition, $b_0 = 0$.

$$\frac{a_m}{R_1^m} + b_m R_1^m = \frac{1}{2\pi}\int_0^{2\pi} \Phi(R_1,\theta) e^{-im\theta} d\theta = \frac{1}{2\pi}\left(-\frac{V_1}{4}\right)\cdot 2\pi\left(\delta_{m,2} + \delta_{m,-2}\right)$$

$$\rightarrow \frac{a_2}{R_1^2} + b_2 R_1^2 = -\frac{V_1}{4} \quad \text{--- ①}$$

$$\frac{a_m}{R_2^m} + b_m R_2 = \frac{1}{2\pi}\int_0^{2\pi} \Phi(R_2,\theta) e^{-im\theta} d\theta = \frac{1}{2\pi}\frac{V_1}{4}\cdot 2\pi\left(\delta_{m,2} + \delta_{m,-2}\right)$$

$$\rightarrow \frac{a_2}{R_2^2} + b_2 R_2^2 = \frac{V_1}{4} \quad \text{--- ②}$$

$① \times R_2^2 - ② \times R_1^2 \Rightarrow$

$$a_2\left(\frac{R_2^2}{R_1^2} - \frac{R_1^2}{R_2^2}\right) = -\frac{V_1}{4}R_2^2 - \frac{V_1}{4}R_1^2$$

$$a_2 = \frac{-\frac{1}{4}\left(R_1^2 V_1 + R_2^2 V_1\right)}{\frac{R_2^2}{R_1^2} - \frac{R_1^2}{R_2^2}} = +\frac{V_1\left(R_1^2 + R_2^2\right)}{4\,\frac{R_1^4 - R_2^4}{R_1^2 R_2^2}} = \underline{\frac{V_1}{4}\cdot\frac{R_1^2 R_2^2}{R_1^2 - R_2^2}}$$

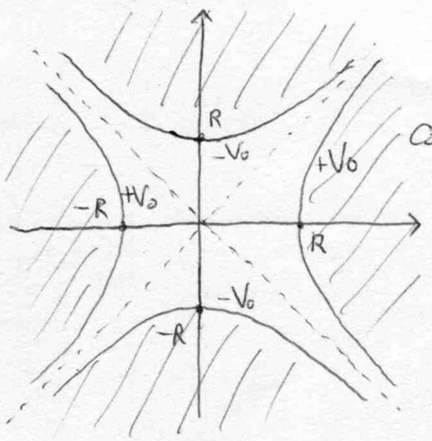
$① \times R_1^2 - ② \times R_2^2$

$$b_2\left(R_1^4 - R_2^4\right) = -\frac{V_1}{4}\left(R_1^2 + R_2^2\right)$$

$$b_2 = \underline{-\frac{V_1}{4\left(R_1^2 - R_2^2\right)}}$$

$$\Rightarrow \underline{\Phi(r,\theta) = \frac{V_1 R_1^2 R_2^2}{4\left(R_1^2 - R_2^2\right)}\frac{2\cos 2\theta}{r^2} - \frac{V_1}{4}\frac{r^2\cdot 2\cos 2\theta}{\left(R_1^2 - R_2^2\right)}}$$

4.



In the regions $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$

$$\Phi(r, \theta) = V_0$$

In the regions $\frac{\pi}{4} < \theta < \frac{3}{4}\pi$ and $\frac{5}{4}\pi < \theta < \frac{7}{4}\pi$

$$\Phi(r, \theta) = -V_0$$

at $|x^2 - y^2| \le R^2$, $\quad \Phi(x, y) = ?$

On the boundary. $x^2 - y^2 = R^2$

$$r_b^2 \cos^2\theta_b - r_b^2 \sin^2\theta_b = r_b^2 \cos 2\theta_b = R^2$$

In the general solution

$$\Phi(r, \theta) = \cancel{b_0}_{\,0} \log\left(\frac{r}{r_0}\right) + \sum_m \left(\frac{a_m}{r^m} e^{im\theta} + b_m r^m e^{im\theta}\right) + c.c.$$

when $r \to 0$, Φ is finite, $a_m = 0$. for all m.,

At the boundary, to have a constant value V_0 or $-V_0$,

$r_b^2 \cos 2\theta_b = R^2$ can be used.

$$\Phi(r_b, \theta_b) = \sum_m b_m r_b^m \left(e^{im\theta_b} + e^{-im\theta_b}\right) = V_0 = V_0 \frac{r_b^2 \cos 2\theta_b}{R^2} \quad \left(\begin{array}{l} -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ \frac{3}{4}\pi < \theta < \frac{5}{4}\pi \end{array}\right.$$

only $m = \pm 2$ left. $\quad b_m = 0$ for $m \neq \pm 2$.

$$\Phi(r, \theta) = \frac{V_0}{R^2} r^2 \cos 2\theta$$

$$\left(\begin{array}{l} \text{when } -\frac{\pi}{4} < \theta < \frac{\pi}{4},\ \frac{3}{4}\pi < \theta < \frac{5}{4}\pi,\quad r_b^2 \cos 2\theta_b = R^2 \Rightarrow \Phi = V_0 \\ \text{when } \frac{\pi}{4} < \theta < \frac{3}{4}\pi,\ \frac{5}{4}\pi < \theta < \frac{7}{4}\pi,\quad r_b^2 \cos 2\theta_b = -R^2 \Rightarrow \Phi = -V_0 \end{array}\right)$$