PHYS 374. Prof Cohen. HW # 8.

1. A cylinder of radius R with charge density $\rho(x) = c x^2 \Theta(R-r)$

a) $x = r\cos\theta$ and $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$$x^2 = r^2\cos^2\theta = r^2 \frac{1}{4}(e^{i\theta}+e^{-i\theta})^2 = \frac{1}{4}r^2(e^{i2\theta} + 2 + e^{-i2\theta})$$

$$\rho(x) = c x^2 \Theta(R-r) = \sum_m \rho_m(r) e^{im\theta}$$

$$\Rightarrow \begin{cases} \rho_0 = \frac{1}{2} c r^2 \Theta(R-r) \\ \rho_{\pm 2} = \frac{1}{4} c r^2 \Theta(R-r) \end{cases}$$

b) $$\Phi_m = \frac{2\pi}{m}\frac{1}{r^m}\int_0^r dr'(r')^{m+1}\rho_m(r') + \frac{2\pi}{m}\int_r^\infty dr'\frac{\rho_m(r')}{(r')^{m+1}}$$

$$\Phi_2 = \frac{2\pi}{2}\frac{1}{r^2}\int_0^r dr'\, r'^3\rho_2(r') + \frac{2\pi}{2}\int_r^\infty \frac{\rho_2(r')}{r^3}dr'$$

when $r > R$, $\quad \Theta(R-r) = 0$

$$\Phi_2 = \pi\frac{1}{r^2}\int_0^R dr'\, r'^3\frac{1}{4}cr'^2 + \frac{\pi}{r^2}\int_R^r dr'\, r'^3\cancel{\left(\frac{1}{4}cr'^2\cdot 0\right)} + \pi\int_r^\infty dr'\cancel{\left(\frac{r'^2c}{4}\cdot 0\right)}\frac{1}{r^3}$$

$$= \frac{\pi}{r^2}\cdot\frac{1}{4}c\,\frac{1}{6}R^6 = \underline{\frac{\pi c}{24}\frac{R^6}{r^2}}$$

$$\Phi_0 = -4\pi\log\left(\frac{r}{r_0}\right)\int_0^R dr'\cdot r'\,\frac{1}{2}cr'^2 - \cancel{\int_R^r dr'\,0} - \cancel{\int_r^\infty dr'\cdot 0}$$

$$= -\frac{4\pi}{2}\log\left(\frac{r}{r_0}\right)\cdot c\,\frac{1}{4}R^4 = \underline{\frac{-\pi c}{2}R^4\log\left(\frac{r}{r_0}\right)}$$

$$\Rightarrow \Phi(r,\theta) = \Phi_0(r) + \Phi_2(r)\,e^{i2\theta}$$

Consider the real part,

$$\underline{\Phi(r,\theta) = -\frac{1}{2}\pi cR^4\log\left(\frac{r}{r_0}\right) + \frac{\pi c}{24}\frac{R^6}{r^2}\cos 2\theta}$$

c) $$\vec{E} = -\nabla\Phi = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right)\Phi$$

$$= -\left[\left(-\frac{1}{2}\pi cR^4\frac{1}{r} + (-2)\frac{\pi c}{24}\frac{R^6}{r^3}\cos 2\theta\right)\hat{r} + \frac{1}{r}\left(-2\frac{\pi c}{24}\frac{R^6}{r^2}\sin 2\theta\right)\hat{\theta}\right]$$

$$= \left(\frac{\pi c}{2}R^4\right)\left(\frac{1}{r} + \frac{R^2}{6r^3}\cos 2\theta\right)\hat{r} + \frac{\pi cR^6}{12r^3}\sin 2\theta\,\hat{\theta}$$

$$= \underline{\frac{\pi cR^4}{2r}\hat{r} + \frac{\pi c}{12r^3}R^6\left(\cos 2\theta\,\hat{r} + \sin 2\theta\,\hat{\theta}\right)}$$

2. $$\Phi_m = \frac{2\pi}{m}\frac{1}{r^m}\int_0^r dr'(r')^{m+1}\rho_m(r') + \frac{2\pi}{m}r^m\int_r^\infty dr'\frac{\rho_m(r')}{(r')^{m-1}}$$

$$\rho_m(r) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\ \rho(r,\theta)e^{im\theta} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\ \frac{1}{2}cr^2(1+\cos 2\theta)e^{im\theta}$$

$$\rho_0(r) = \frac{1}{\cancel{2\pi}}\cdot\frac{1}{2}cr^2\cdot\cancel{2\pi} = \frac{1}{2}cr^2$$

$$\rho_2(r) = \frac{1}{2\pi}\frac{1}{2}cr^2\int_0^{2\pi} d\theta\ (1+\cos 2\theta)e^{i2\theta} = \frac{1}{2\pi}\frac{1}{2}cr^2\int_0^{2\pi}(\cancel{1}+\cos 2\theta)(\cos 2\theta + i\cancel{\sin 2\theta})\,d\theta$$

$$= \frac{1}{\cancel{2\pi}}\cdot\frac{1}{2}cr^2\cdot\frac{1}{2}\cancel{2\pi} = \frac{1}{4}cr^2$$

$$\Phi_2 = \frac{2\pi}{2}\frac{1}{r^2}\int_0^r dr'(r')^3\rho_2(r') + \frac{2\pi}{2}r^2\int_r^\infty dr'\frac{\rho_2(r')}{r'^1}\Theta(R-r)$$

$$= \frac{\pi}{r^2}\int_0^r dr'\ \frac{1}{4}cr'^5 + \pi r^2\int_r^R \frac{1}{\cancel{r'}}\frac{1}{4}cr'^{\cancel{2}}$$

$$= \frac{\pi}{\cancel{r^2}}c\cdot\frac{r^{\cancel{6}\,4}}{24} + \pi r^2\frac{c}{4}\left(\frac{1}{2}R^2 - \frac{1}{2}r^2\right)$$

$$= \frac{\pi}{8}cr^4\left(\frac{1}{3}-1\right) + \frac{\pi}{8}cR^2r^2 = \underline{\frac{\pi}{8}cr^2\left(R^2 - \frac{2}{3}r^2\right)}$$

$$\Phi_0 = -4\pi\log\frac{r}{r_0}\int_0^r dr'r'\rho_0(r) - 4\pi\int_r^\infty dr'\left(\log\frac{r'}{r_0}\right)r'\rho_0(r')$$

$$= -4\pi\log\frac{r}{r_0}\int_0^r dr'\cdot r'\frac{1}{2}cr'^2 - 4\pi\int_r^R dr'\ r'\log\frac{r'}{r_0}\frac{1}{2}cr'^2$$

$$= -4\pi\log\frac{r}{r_0}\frac{1}{2}c\ \frac{1}{4}r^4 - 4\pi\frac{1}{2}c\left[\left[\log\frac{r'}{r_0}\cdot\frac{1}{4}r'^4\right]_r^R - \int_r^R\frac{1}{\cancel{r'}}\frac{r'^{\cancel{4}\,3}}{4}dr'\right]$$

$$= -4\pi\frac{1}{8}cr^4\log\frac{r}{r_0} - \frac{c}{2}4\pi\left(\log\left(\frac{R}{r_0}\right)R^4\frac{1}{4} - \frac{1}{4}r^4\log\frac{r}{r_0} - \frac{1}{16}(R^4-r^4)\right)$$

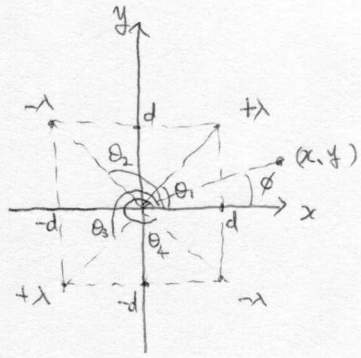
$$= \cancel{-\frac{\pi}{2}cr^4\log\frac{r}{r_0}} - \frac{2\pi}{4}c\log\left(\frac{R}{r_0}\right)R^4 + \cancel{\frac{\pi}{2}cr^4\log\frac{r}{r_0}} + \frac{4\pi}{32}c(R^4-r^4)$$

$$= \underline{\frac{\pi}{8}c(R^4-r^4) - \frac{\pi}{2}cR^4\log\left(\frac{R}{r_0}\right)}$$

$$\therefore\ \Phi = \Phi_0 + \Phi_2\cos 2\theta$$

$$= \underline{\frac{\pi}{8}c\left(R^4 + \underset{-r^2}{\wedge}\left(R^2r^2 - \frac{2}{3}r^4\right)\cos 2\theta\right) - \frac{\pi}{2}cR^4\log\frac{R}{r_0}}$$

3.



$$\Phi = \sum_i \lambda_i \left[-2 \log\left(\frac{|\vec{r} - \vec{r}_i|}{r_0} \right) \right]$$

$$r_1^2 = r_2^2 = r_3^2 = r_4^2 = 2d^2$$

$$|r - r_i|^2 = r^2 + r_i^2 - 2 r r_i \cos(\phi - \theta_i)$$

a)

$$\Phi = -2\lambda \left[\log\left(\frac{1}{r_0}\sqrt{r^2+2d^2-2\sqrt{2}dr\cos(\phi-\tfrac{\pi}{4})} \right) - \log\left(\frac{1}{r_0}\sqrt{r^2+2d^2-2\sqrt{2}dr\cos(\phi-\tfrac{3}{4}\pi)} \right) \right.$$

$$\left. + \log\left(\frac{1}{r_0}\sqrt{r^2+2d^2-2\sqrt{2}dr\cos(\phi-\tfrac{5}{4}\pi)} \right) - \log\left(\frac{1}{r_0}\sqrt{r^2+2d^2-2\sqrt{2}dr\cos(\phi-\tfrac{7}{4}\pi)} \right) \right]$$

$$= -2\lambda \log \left[\frac{\sqrt{r^2+2d^2-2\sqrt{2}dr\cos(\phi-\frac{\pi}{4})}\sqrt{r^2+2d^2-2\sqrt{2}dr\cos(\phi-\frac{5}{4}\pi)}}{\sqrt{r^2+2d^2-2\sqrt{2}dr\cos(\phi-\frac{3}{4}\pi)}\sqrt{r^2+2d^2-2\sqrt{2}dr\cos(\phi-\frac{7}{4}\pi)}} \right]$$

b) total charge is zero $\Rightarrow$ $\underline{b_0 = 0}$

$$a_m = \frac{1}{m}\sum_{i=1}^{4} \lambda_i e^{im\theta_i} r_i^m$$

$$= \frac{1}{m}(\sqrt{2}d)^m \lambda \left(e^{im\frac{\pi}{4}} - e^{im\frac{3}{4}\pi} + e^{im\frac{5}{4}\pi} - e^{im\frac{7}{4}\pi} \right)$$

c)

$$a_1 = \sqrt{2}d\lambda \left(e^{i\frac{\pi}{4}} - e^{i\frac{3}{4}\pi} + e^{i\frac{5}{4}\pi} - e^{i\frac{7}{4}\pi} \right)$$

$$= \sqrt{2}d\lambda \frac{1}{\sqrt{2}} \left(\cancel{1} + \cancel{i} - (-\cancel{1} + \cancel{i}) + (-\cancel{1} - \cancel{i}) - (\cancel{1} - \cancel{i}) \right) = \underline{0}$$

$$a_2 = \frac{1}{2} 2 d^2 \lambda \left(\underbrace{e^{i\frac{\pi}{2}}}_{i} - \underbrace{e^{i\frac{3}{2}\pi}}_{-i} + \underbrace{e^{i\frac{5}{2}\pi}}_{i} - \underbrace{e^{i\frac{7}{2}\pi}}_{-i} \right) = \underline{4 i d^2 \lambda}$$

$$a_3 = \frac{1}{3} 2^{3/2} d^3 \lambda \left(e^{i\frac{3}{4}\pi} - e^{i\frac{9}{4}\pi} + e^{i\frac{15}{4}\pi} - e^{i\frac{21}{4}\pi} \right)$$

$$= \frac{2\sqrt{2}}{3} d^3 \lambda \frac{1}{\sqrt{2}} \left(-\cancel{1} + \cancel{i} - (\cancel{1} + \cancel{i}) + (\cancel{1} - \cancel{i}) - (-\cancel{1} - \cancel{i}) \right) = \underline{0}$$

$$a_4 = \frac{1}{4} 4 d^4 \lambda \left(\cancel{e^{i\pi}} - \cancel{e^{i3\pi}} + \cancel{e^{i5\pi}} - \cancel{e^{i7\pi}} \right) = \underline{0}$$

$$\Phi(r,\phi) = \sum \frac{a_m}{r^m} e^{im\varphi} + \frac{a_m^*}{r^m} e^{-im\varphi} \simeq \underline{8\left(\frac{d}{r}\right)^2 \lambda \sin 2\varphi}$$

3. (d) Set d=1 and λ=1. Φ is the exact function and App[r] is a function truncated at m=2, 3 and 4.

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d = 1; λ = 1; ϕ = π/4;
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$$\Phi[r_] = -2\,\lambda\,\mathrm{Log}\left[\sqrt{\left(\frac{\left(r^2+2\,d^2-2\,d\,\sqrt{2}\,r*\mathrm{Cos}[\phi-\frac{\pi}{4}]\right)\left(r^2+2\,d^2-2\,d\,\sqrt{2}\,r*\mathrm{Cos}[\phi-\frac{5\pi}{4}]\right)}{\left(r^2+2\,d^2-2\,d\,\sqrt{2}\,r*\mathrm{Cos}[\phi-\frac{3\pi}{4}]\right)\left(r^2+2\,d^2-2\,d\,\sqrt{2}\,r*\mathrm{Cos}[\phi-\frac{7\pi}{4}]\right)}\right)}\right]$$

$$-2\,\mathrm{Log}\left[\sqrt{\frac{\left(2+r^2-2\,\sqrt{2}\,r\,\mathrm{Cos}[\frac{\pi}{4}-\phi]\right)\left(2+r^2-2\,\sqrt{2}\,r\,\mathrm{Cos}[\frac{5\pi}{4}-\phi]\right)}{\left(2+r^2-2\,\sqrt{2}\,r\,\mathrm{Cos}[\frac{3\pi}{4}-\phi]\right)\left(2+r^2-2\,\sqrt{2}\,r\,\mathrm{Cos}[\frac{7\pi}{4}-\phi]\right)}}\right]$$

$$\mathrm{App}[r_] = \frac{8\,d^2}{r^2}\,\mathrm{Sin}[2*\phi]$$

$$\frac{8\,\mathrm{Sin}[2\,\phi]}{r^2}$$

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Plot [ {Φ[r], App[r]} , {r, 0, 10}, PlotRange → {0, 15}]
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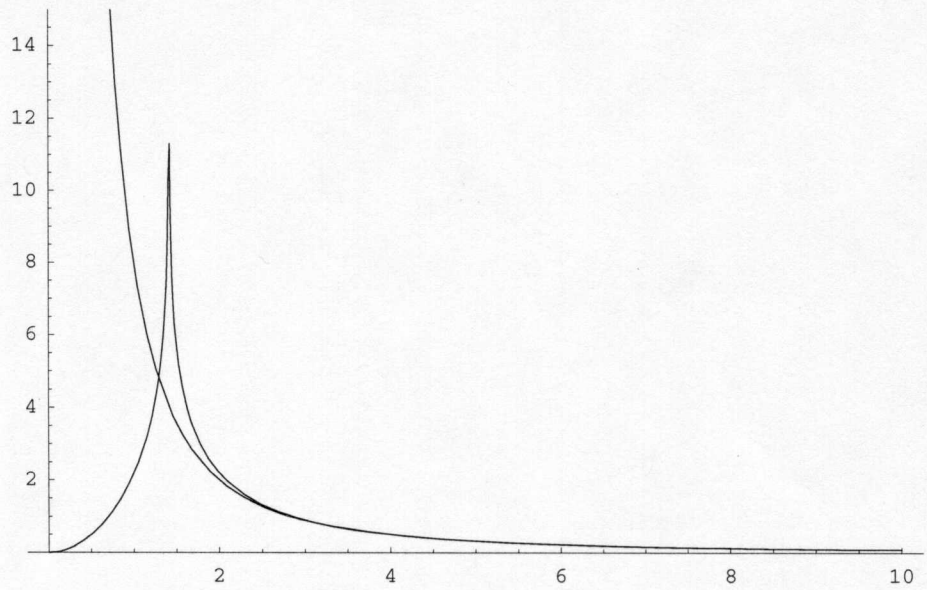
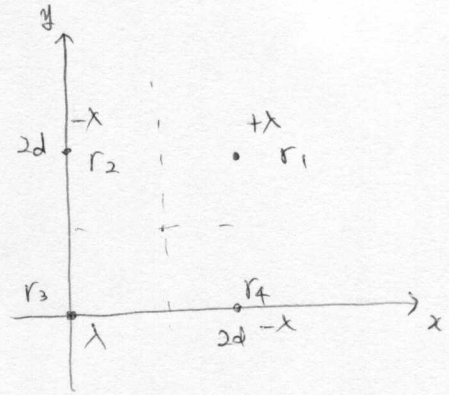


- Graphics -

4.



$|\vec{r}_1| = 2\sqrt{2}\,d$. $\quad |\vec{r}_2| = |\vec{r}_4| = 2d$. $|\vec{r}_3| = 0$

$\theta_1 = \frac{\pi}{4}$ $\quad \theta_2 = \frac{\pi}{2}$ $\quad \theta_4 = 0$

$$a_m = \frac{1}{m}\sum_i \lambda_i e^{im\theta_i} r_i^{\,m}$$

for $m=2$,

$$a_2' = \frac{1}{2}\left(\lambda e^{i2\frac{\pi}{4}}(2\sqrt{2}d)^2 - \lambda e^{i2\frac{\pi}{2}}(2d)^2 + \lambda\cdot 0 - \lambda e^{i2\cdot 0}(2d)^2\right)$$

$$= \frac{1}{2}\lambda d^2\left(8i - (-\cancel{4}) - \cancel{4}\right) = \underline{4i\lambda d^2}$$

$$a_2' = \underbrace{a_2}_{\text{from } 3.(c)}$$