

$$1. \quad \Lambda_x = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Lambda_y = \begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a) $\Lambda_x \Lambda_y = \begin{pmatrix} \gamma^2 & -\beta\gamma & -\beta\gamma^2 & 0 \\ -\beta\gamma^2 & \gamma & \beta^2\gamma^2 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\Lambda_y \Lambda_x = \begin{pmatrix} \gamma^2 & -\beta\gamma^2 & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ -\beta\gamma^2 & \beta^2\gamma^2 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Lambda_x \Lambda_y \neq \Lambda_y \Lambda_x$$

b) $\vec{x}^4 = \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Lambda_x \Lambda_y \vec{x}^4 = \begin{pmatrix} \gamma^2 t \\ -\beta\gamma^2 t \\ -\beta\gamma t \\ 0 \end{pmatrix}$$

c) In new frame $\vec{x}''^4 = \begin{pmatrix} t'' \\ x'' \\ y'' \\ z'' \end{pmatrix} = \Lambda_x \Lambda_y \vec{x}^4 = \begin{pmatrix} \gamma^2 t \\ -\beta\gamma^2 t \\ -\beta\gamma t \\ 0 \end{pmatrix}$

$$\Rightarrow t'' = \gamma^2 t, \quad x'' = -\beta\gamma^2 t = -\beta t''$$

$$z'' = 0 \quad y'' = -\beta\gamma t = -\frac{\beta}{\gamma} t''$$

$$\therefore x''(t') = -\beta t''$$

$$\underline{y''(t'') = -\frac{\beta}{\gamma} t''}$$

2. a) The particle at rest at the origin remain at rest in the moving frame (new frame)

$$\vec{v}_{\text{frame}} + \frac{d\vec{x}''}{dt''} = 0 \quad \vec{v}_{\text{frame}} = -\frac{d\vec{x}''}{dt''}$$

$$\frac{d\vec{x}''}{dt''} = \begin{pmatrix} \frac{dx''}{dt''} \\ \frac{dy''}{dt''} \\ \frac{dz''}{dt''} \end{pmatrix} = \begin{pmatrix} -\beta \\ -\frac{\beta}{\gamma} \\ 0 \end{pmatrix}$$

$$\therefore \vec{v}_{\text{frame}} = \begin{pmatrix} \beta \\ \beta/\gamma \\ 0 \end{pmatrix}$$

b)

$$\gamma = (1 - \beta^2)^{-\frac{1}{2}}$$

for a small β , $\beta^2 \sim 0$

then,

$$\vec{v} \rightarrow \begin{pmatrix} +\beta \\ \beta \\ 0 \end{pmatrix}$$

∴ Angle approaches 45°

c) $|\vec{v}_{\text{frame}}| = \left(\beta^2 + \left(\frac{\beta}{\gamma}\right)^2 \right)^{\frac{1}{2}}$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \gamma^2(1-\beta^2) = 1 \quad \frac{1}{\gamma^2} = 1 - \beta^2$$

$$|\vec{v}_{\text{frame}}| = \left[\beta^2 + \beta^2(1-\beta^2) \right]^{\frac{1}{2}} = \sqrt{2\beta^2 - \beta^4}$$

when $\beta^2 > 2$, $2\beta^2 - \beta^4 < 0 \rightarrow |\vec{v}_{\text{frame}}| \text{ is imaginary}$
 $\therefore \beta^2 \leq 2 \rightarrow \beta^2 - 1 \leq 1$ \rightarrow meaning less!

$$|\vec{v}_{\text{frame}}| = \sqrt{2\beta^2 - \beta^4} = \sqrt{1 - 1 + 2\beta^2 - \beta^4} = \sqrt{1 - (1 - \beta^2)^2} < 1$$

Q.E.D.

3. length of the train : $l_{tr} = 0.5 \text{ km}$. velocity of the train $v = \frac{4}{5}c$
 $\Rightarrow \beta = \frac{4}{5}, \gamma = \frac{5}{3}$

length of the tunnel : $L_{tun} = 0.6 \text{ km}$

at time t_1 , the front of train leaves the tunnel, $\begin{pmatrix} t_1 \\ L_{tun} \\ 0 \end{pmatrix} = \vec{x}_{front}$

at time t_2 , the rear of train enters the tunnel $\begin{pmatrix} t_2 \\ 0 \\ 0 \end{pmatrix} = \vec{x}_{rear}$

$$t_1 = \frac{0.6 \text{ km}}{\frac{4}{5}c} = \frac{3}{4c}, \quad t_2 = \frac{\frac{3}{5}0.5 \text{ km}}{\frac{4}{5}c} = \frac{3}{8c}$$

$$\Rightarrow \vec{x}_{front} = \begin{pmatrix} \frac{3}{4} \text{ km} \\ 0.6 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}_{rear} = \begin{pmatrix} \frac{3}{8} \text{ km} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c = 1$$

(b) by the Lorentz transformation, $\Lambda_x = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\vec{x}'_{front} = \Lambda_x \vec{x}_{front} = \begin{pmatrix} t_1\gamma - \beta\gamma L_{tun} \\ -\beta\gamma t_1 + \gamma L_{tun} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (\frac{3}{4} - \frac{4}{5} \cdot \frac{3}{8}) \\ -1 + 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.45 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} t'_f \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{x}'_{rear} = \Lambda_x \vec{x}_{rear} = \begin{pmatrix} t_2\gamma \\ -\beta\gamma t_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{8} \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} t'_r \\ x'_{rear} \\ 0 \\ 0 \end{pmatrix}$$

$$t'_f < t'_r$$

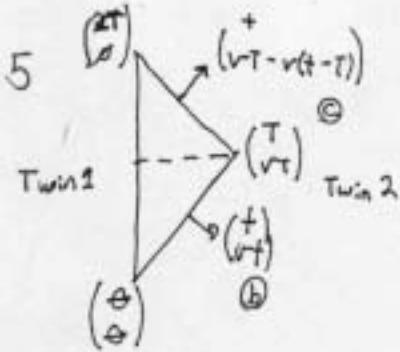
$$4. \tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

In the rest frame of tunnel,

$$\tau_{\text{tun}}^2 = \left(\frac{3}{4} - \frac{5}{8}\right)^2 - (0.6)^2 = \underline{-0.219375}$$

In the moving frame with train

$$\tau_{\text{train}}^2 = \left(0.45 - \frac{5}{8}\right)^2 - \left(-\frac{1}{2}\right)^2 = \underline{-0.219375}$$



Twin 1

$$x_i = 0 \quad x_f = 0$$

$$v = 0$$

$$\tilde{r} = \int_0^{2T} dt'$$

$$\tilde{L} = 2T$$

b1

Twin 2

$$x_i = vt$$

$$\dot{x} = v \quad \ddot{x} = -v$$

$$\tilde{r} = \int_0^T \sqrt{1-v^2} dt' + \int_T^{2T} \sqrt{1-v^2} dt'$$

$$\tilde{L} = \frac{T}{\gamma} + \frac{2T}{\gamma} - \frac{T}{\gamma} = \frac{2T}{\gamma}$$

$$\textcircled{a} \quad \begin{pmatrix} \gamma - \beta\gamma \\ -\beta\gamma \end{pmatrix} \begin{pmatrix} + \\ 0 \end{pmatrix} = \begin{pmatrix} +\gamma \\ -\beta + \gamma \end{pmatrix}$$

$$\textcircled{b} \quad \begin{pmatrix} \gamma - \beta\gamma \\ -\beta\gamma \end{pmatrix} \begin{pmatrix} + \\ vT - v(t-T) \end{pmatrix} = \begin{pmatrix} +\gamma - \beta\gamma vT + \beta\gamma v(t-T) \\ -\beta\gamma + \gamma vT - \gamma v(t-T) \end{pmatrix}$$

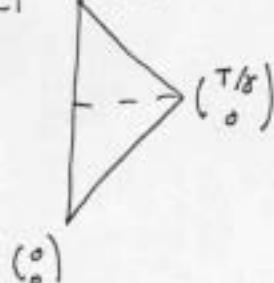
$$\textcircled{c} \quad \begin{pmatrix} \gamma - \beta\gamma \\ -\beta\gamma \end{pmatrix} \begin{pmatrix} + \\ vt \end{pmatrix} = \begin{pmatrix} +/\gamma \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \gamma + -\beta\gamma(2vT - vt) \\ -\beta\gamma t - \gamma(2vT - vt) \end{pmatrix}$$

c1

$$\begin{pmatrix} 2\gamma T \\ -2\beta\gamma T \end{pmatrix}$$

$$\Delta V = \frac{\Delta d}{\Delta t}$$



Twin 1

$$\tilde{r} = \int_0^{2\gamma T} \sqrt{1 - \left(\frac{-2\beta\gamma T}{2\gamma T}\right)^2} dt'$$

$$\tilde{L} = \int_0^{2\gamma T} \sqrt{1 - (-\beta)^2} dt'$$

$$\tilde{L} = \sqrt{1 - v^2} 2\gamma T$$

$$\tilde{L} = 2T$$

$$\begin{aligned}
 & \frac{T_{\text{min}}}{2} \\
 L &= \int_0^{\frac{T}{\gamma}} \sqrt{1-\delta} dt' + \int_{\frac{T}{\gamma}}^{2\gamma T} \sqrt{1 - \frac{4v^2}{(1+v^2)^2}} dt' \\
 &+ \int_{\frac{T}{\gamma}}^{2\gamma T} \sqrt{\frac{1+2v^2+v^4-4v^2}{(1+v^2)^2}} dt' \\
 &+ \int_{\frac{T}{\gamma}}^{2\gamma T} \sqrt{\frac{(1-v^2)^2}{(1+v^2)^2}} dt' \\
 &+ \int_{\frac{T}{\gamma}}^{2\gamma T} \frac{(1-v^2)}{(1+v^2)} dt' \\
 &+ \frac{(1-v^2)}{(1+v^2)} (2\gamma T - \frac{T}{\gamma}) \\
 &+ \frac{2T\delta(1-v^2)}{(1+v^2)} - \frac{T}{\gamma} \frac{(1-v^2)}{(1+v^2)} \\
 &+ \frac{2T - T(1-v^2)}{\delta(1+v^2)} \\
 &+ \frac{T(2-1+v^2)}{\delta(1+v^2)} \\
 &+ \frac{T(1+v^2)}{\delta(1+v^2)} \\
 \frac{T}{\gamma} &+ \frac{T}{\gamma} = \frac{2T}{\delta}
 \end{aligned}$$