

## Physics 374 Homework 5

- 1) In this problem, I want you to work out the effect of making two Lorentz transformations in perpendicular directions (x and y) where we will assume that the magnitudes of each are the same

$$\Lambda_x = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Lambda_y = \begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- a) Find the matrix for the transformations for a boost in the y direction followed by a boost in x and the matrix transformations for a boost in the x direction followed by a boost in y. If you do this correctly you will find that the two matrices are different. This reflects the fact the Lorentz group is non-Abelian (for you Math junkies)
- b) Consider the space-time trajectory for a particle at rest at origin in the original frame:

$$\vec{x}^4 = \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Find the four vector in the frame of a boost in y followed by a boost in x (which is given by  $\Lambda_x \Lambda_y$ ).

- c) Show that in this new frame (denoted the double prime frame) the position of this particle

$$x''(t'') = -\beta t'' \quad y''(t'') = -\frac{\beta t''}{\gamma}$$

- 2) In this problem we will use the results of problem 1 to determine the velocity of the new frame
- a) From 1c) show that the velocity of the new frame relative to the old one is given by the (three

vector)  $\vec{v} = \begin{pmatrix} \beta \\ \beta/\gamma \\ 0 \end{pmatrix}$  which is not at  $45^\circ$ .

- b) Verify that at small  $\beta$  the angle approaches  $45^\circ$  as in Newtonian physics.
- c) Show that the speed of the new frame is less than the speed of light, *i.e.* show the magnitude of the velocity in part a) is less than unity.

- 3) In this problem I want to work through an inverse version of the “pole vaulter” problem discussed in class. Here, I want to consider a train of length .5 km (in its rest frame) that is moving with a velocity of  $4/5 c$  (which corresponds to a  $\gamma$  factor of  $5/3$ ) along a straight track toward a tunnel that has a length of .6 km (in its rest frame). Clearly, at rest the train fits in the tunnel, and given the fact that in the rest frame of the tunnel the train is Lorentz contracted by a factor of  $3/5$  it clearly will fit in the tunnel. However, in a frame comoving with the train, the tunnel is Lorentz contracted by a factor of  $3/5$  and the train will not fit in. This paradox is resolved through the relativity of simultaneity. To show this

- a) Find the space-time points in the rest frame of the tunnel describing the events:

- i) The front of the train leaves the tunnel
- ii) The rear of the train enters the tunnel

- b) Next make a Lorentz transformation to a frame co-moving with the train and describe the same two space-time points. Explain how this resolves the paradox.
- 4) For the two space-time events in problem calculate the proper time interval  $\tau^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$  for the rest frame of the tunnel and for the frame comoving with the train. Verify that this interval is the same in both frames. (If it is not you screwed up---go back and check your work!!!)
- 5) In class we worked through the twin paradox problem in the frame of twin 1, the twin who stayed home. Here I want you to work through the same problem in the frame co-moving with twin 2 on her way out. In the rest frame of twin 1 the trajectories of the two twins are  $x_1 = 0$ ,

$$\bar{x}_2 = v t \theta\left(\frac{T}{2} - t\right) \theta(t) + \theta\left(t - \frac{T}{2}\right) \theta(T - t) \left(\frac{v T}{2} - v \left(t - \left(\frac{T}{2}\right)\right)\right) \text{ where } \theta \text{ is the usual step}$$

function, which is zero for negative arguments and 1 for positive arguments. Note that the form given for the trajectories is not explicitly covariant in terms of Lorentz transformations.

- a) Use the expression  $\tau = \int dt \sqrt{1-v^2}$  to find the proper time of each of the two twins from the event where they leave each other to the event where they meet again. Do this calculation in the rest frame of twin 1. (This just redoes what we did in class using these parameters). You should find that Twin 1 is older by a factor of  $\gamma$ .
- b) Find the trajectories in the frame comoving with twin 2 on his outward voyage.
- c) Working in this new frame use the expression  $\tau = \int dt \sqrt{1-v^2}$  to find the proper time of the two twins. You should find your result identical to those of part a).