

1. $U(x) = \frac{U_0}{1 + (\frac{x}{L})^2}$ with $U_0 > 0$

a) $\frac{1}{1+x} \simeq 1 - x - x^2 + \dots$

$$U(x) \simeq U_0 (1 - (\frac{x}{L})^2)$$

$$F(x) = -\frac{\partial U}{\partial x} = -U_0 \left(-\frac{2x}{L^2} \right) = \frac{2xU_0}{L^2} = m\ddot{x}$$

$$\ddot{x} - \frac{2U_0}{mL^2} x = 0$$

$$x = A \cosh(\omega_0 t) + B \sinh(\omega_0 t) \quad \text{where } \omega_0^2 = \frac{2U_0}{mL^2}$$

Using the initial conditions, $\dot{x}(0) = 0$ $x(0) = h$

$$\dot{x} = \omega_0 A \sinh(0) + \omega_0 B \cosh(0) = 0 \rightarrow B = 0$$

$$x(0) = A \cosh(0) = X_0 \rightarrow A = X_0$$

$$x(t) = L \cosh \omega_0 t \rightarrow \omega_0 t = \cosh^{-1} \left(\frac{x(t)}{X_0} \right)$$

$$\Rightarrow t = \frac{1}{\sqrt{\frac{2U_0}{mL^2}}} \cosh^{-1} \left(\frac{x}{X_0} \right)$$

when $\frac{x}{L} = \frac{1}{2}$, $t = \frac{1}{\sqrt{\frac{2U_0}{mL^2}}} \cosh^{-1} \left(\frac{h}{2X_0} \right)$

b) "Very near top" $\frac{x_0}{L} \ll 1$

$x_0 \ll L$

$$2.a) E = U(x_0) = \frac{U_0}{1 + \left(\frac{x_0}{L}\right)^2}$$

$$\frac{1}{2} m \dot{x}^2 + \frac{U_0}{1 + \left(\frac{x}{L}\right)^2} = \frac{U_0}{1 + \left(\frac{x_0}{L}\right)^2}$$

$$\dot{x}^2 = \frac{\frac{2U_0}{m}}{1 + \left(\frac{x_0}{L}\right)^2} - \frac{\frac{2U_0}{m}}{1 + \left(\frac{x}{L}\right)^2} = \frac{2U_0}{m} \left[\frac{1}{1 + \left(\frac{x_0}{L}\right)^2} - \frac{1}{1 + \left(\frac{x}{L}\right)^2} \right]$$

$$\frac{dx}{dt} = \sqrt{\frac{2U_0}{m}} \left[\frac{1}{1 + \left(\frac{x_0}{L}\right)^2} - \frac{1}{1 + \left(\frac{x}{L}\right)^2} \right]^{\frac{1}{2}}$$

$$\sqrt{\frac{m}{2U_0}} \int_{x_0}^{x_2} \left[\frac{1}{1 + \left(\frac{x_0}{L}\right)^2} - \frac{1}{1 + \left(\frac{x}{L}\right)^2} \right]^{-\frac{1}{2}} dx = \int_0^T dt$$

$$T = \sqrt{\frac{m}{2U_0}} \int_{x_0}^{x_2} \left[\frac{1}{1 + \left(\frac{x_0}{L}\right)^2} - \frac{1}{1 + \left(\frac{x}{L}\right)^2} \right]^{-\frac{1}{2}} dx$$

$$b) \frac{1}{1 + \left(\frac{x}{L}\right)^2} \approx 1 - \left(\frac{x}{L}\right)^2$$

$$\frac{1}{1 + \left(\frac{x}{L}\right)^2} \approx 1 - \left(\frac{x}{L}\right)^2$$

$$\frac{x_0}{L} = 10^{-4}$$

$$\left[\frac{1}{1 + \left(\frac{x_0}{L}\right)^2} - \frac{1}{1 + \left(\frac{x}{L}\right)^2} \right]^{-\frac{1}{2}} \approx \left(1 - \left(\frac{x_0}{L}\right)^2 - 1 + \left(\frac{x}{L}\right)^2 \right)^{-\frac{1}{2}} \approx \left(\frac{x}{L}\right)^{-1}$$

$$T = \sqrt{\frac{m}{2U_0}} \int_{x_0}^{x_2} \left(\frac{L}{x}\right) dx = \sqrt{\frac{m}{2U_0}} L \log\left(\frac{x_2}{x_0}\right) = \sqrt{\frac{m}{2U_0}} L \log\left(\frac{1}{2 \cdot 10^{-4}}\right)$$

$$\therefore T = \sqrt{\frac{m}{2U_0}} L \log\left(\frac{10^4}{2}\right) \approx \sqrt{\frac{m}{2U_0}} L \times 8.52.$$

from a 1)

$$t = \sqrt{\frac{m}{2U_0}} L \operatorname{Cosh}^{-1}\left(\frac{L}{2x_0}\right) \approx \sqrt{\frac{m}{2U_0}} L \times 9.21$$

3. Lorentz transformation.

$$\left. \begin{array}{l} t' = t\gamma - \beta\gamma x \\ x' = -\beta\gamma t + \gamma x \\ y' = y \\ z' = z \end{array} \right\} \quad \begin{aligned} \gamma &= \frac{1}{\sqrt{1-\beta^2}} & \gamma^2 &= \frac{1}{1-\beta^2} \\ && \Rightarrow \gamma^2(1-\beta^2) &= 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow t'^2 - x'^2 - y'^2 - z'^2 &= (t\gamma - \beta\gamma x)^2 - (-\beta\gamma t + \gamma x)^2 = y^2 - z^2 \\ &= t^2\gamma^2 - \cancel{2t\beta\gamma^2x} + \beta^2\gamma^2x^2 - \cancel{\beta^2\gamma^2t^2} + \cancel{2\beta\gamma^2tx} - \cancel{\gamma^2x^2} - y^2 - z^2 \\ &= t^2(\underbrace{\gamma^2 - \beta^2\gamma^2}_{1}) + (\underbrace{\beta^2\gamma^2 - \gamma^2}_{-1})x^2 - y^2 - z^2 \\ &= t^2 - x^2 - y^2 - z^2 \end{aligned}$$

$\xrightarrow{\text{invariant}}$

4.

$$\beta_1 = \frac{v_1}{c}, \quad \beta_2 = \frac{v_2}{c}, \quad \beta_{\text{tot}} = \frac{v_{\text{tot}}}{c}$$

$$\begin{pmatrix} t_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} t_0 \\ x_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} t_0 \\ x_0 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} t_2 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} t_0 \\ x_0 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & \gamma_1\gamma_2(-\beta_1-\beta_2) \\ \gamma_1\gamma_2(-\beta_1-\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) \end{pmatrix} \begin{pmatrix} t_0 \\ x_0 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \gamma_{\text{tot}} & -\beta_{\text{tot}}\gamma_{\text{tot}} \\ -\beta_{\text{tot}}\gamma_{\text{tot}} & \gamma_{\text{tot}} \end{pmatrix} \begin{pmatrix} t_0 \\ x_0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} \gamma_{\text{tot}} = \gamma_1\gamma_2(1+\beta_1\beta_2) & \text{--- ①} \\ +\beta_{\text{tot}}\gamma_{\text{tot}} = +\gamma_1\gamma_2(\beta_1+\beta_2) & \text{--- ②} \end{cases}$$

$$\textcircled{①} \quad \beta_{\text{tot}} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}$$

$$\Rightarrow v_{\text{tot}} = \frac{v_1 + v_2}{1 + v_1 v_2}$$