

PHYS 374. Homework #3.

1) $f(t) = A \cos^3(\omega t)$

$$= A \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right]^3 = \frac{A}{8} \left(e^{3i\omega t} + 3e^{i\omega t} + 3e^{-i\omega t} + e^{-3i\omega t} \right)$$

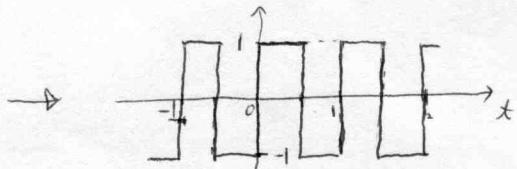
$$= \sum_{n=-\infty}^{\infty} c_n e^{-in\omega t}$$

$$c_1 = c_{-1} = 3 \cdot \frac{A}{8}, \quad c_3 = c_{-3} = A \cdot \frac{1}{8}$$

$$c_2 = c_{-2} = c_4 = c_{-4} = \dots = 0$$

for $|n| > 3$, $c_n = 0$.

2) $\textcircled{2} \quad f(t) = \begin{cases} +1 & \text{for } 0 \leq \text{mod}_1(t) \leq \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \leq \text{mod}_1(t) < 1 \end{cases}$



$$\underline{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi nt}}$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{i2\pi nt} dt = \sum_{n=-\infty}^{\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} c_n e^{2\pi i(m-n)t} dt$$

when $m \neq n$,

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} c_n e^{2\pi i(m-n)t} dt = c_n \frac{e^{2\pi i(m-n)t}}{2\pi(m-n)i} \Big|_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= c_n \frac{1}{2\pi(m-n)i} \left(e^{\pi i(m-n)T} - e^{-\pi i(m-n)T} \right) = c_n \frac{\cancel{\pi} \sin(m-n)\pi T}{\pi \cancel{\pi}(m-n)} = 0 \quad (\because \text{integer and } T=1)$$

when $m = n$,

$$c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{2\pi i(m-n)t} dt = c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} dt = c_n T$$

Then, we have, $c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{i2\pi nt} dt$

$$\begin{aligned}
 C_n &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{i2\pi nt} dt = \int_{-\frac{T}{2}}^0 (-1) e^{i2\pi nt} dt + \int_0^{\frac{T}{2}} e^{i2\pi nt} dt \\
 &= -\frac{e^{i2\pi nt}}{i2\pi n} \Big|_{-\frac{T}{2}}^0 + \frac{e^{i2\pi nt}}{i2\pi n} \Big|_0^{\frac{T}{2}} \\
 &= -\frac{1}{i2\pi n} + \frac{e^{i2\pi nt}}{i2\pi n} + \frac{e^{i\pi n}}{i2\pi n} - \frac{1}{i2\pi n} \\
 &= \frac{1}{i2\pi n} (2e^{i\pi n} - 2)
 \end{aligned}$$

when n is even, $e^{i\pi n} = 1 \rightarrow C_n = 0$

when n is odd, $e^{i\pi n} = -1 \rightarrow C_n = \frac{-4}{i2\pi n} = \frac{2i}{n\pi}$

Note I think there is misused \leftarrow sign.

When $f(t)$ is defined by $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{+i2\pi nt}$, $c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-i2\pi nt} dt$
 the answer will be $c_n = -\frac{2i}{n\pi}$ for n odd
 When $f(t)$ is defined by $f(t) = \sum_{n=-\infty}^{\infty} e^{-i2\pi nt}$, then $c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{i2\pi nt} dt$
 the answer will be $c_n = +\frac{2i}{n\pi}$ for n odd.

b) Let's think $C_n = -\frac{2i}{n\pi}$

$$C_n^* = \frac{2i}{n\pi} \quad C_{-n} = \frac{2i}{n\pi} \quad \Rightarrow \quad \underline{C_{-n} = C_n^*}$$

3. a) $\frac{d}{dt} \left(\frac{v}{v_c} \right)_0 = -\frac{1}{\tau} \left(\frac{v}{v_c} \right)_0$ with initial velocity $\left(\frac{v}{v_c} \right)_0$.

$$\frac{1}{\left(\frac{v}{v_c} \right)_0} \frac{d}{dt} \left(\frac{v}{v_c} \right)_0 = -\frac{1}{\tau} \quad \int_{v_1}^v \frac{1}{\left(\frac{v}{v_c} \right)_0} d \left(\frac{v}{v_c} \right)_0 = - \int_0^t \frac{1}{\tau} dt$$

$$\text{Log} \left[\frac{v}{v_c} \right] - \text{Log} \left[\frac{v_i}{v_c} \right] = -\frac{t}{\tau} \quad \left(\frac{v}{v_c} \right)_0 / \left(\frac{v_i}{v_c} \right)_0 = e^{-\frac{t}{\tau}}$$

$$\text{When } t = 0, \quad \left(\frac{v}{v_c} \right)_0 / \left(\frac{v_i}{v_c} \right)_0 = 1 = \left(\frac{v}{v_c} \right)_0$$

$$\text{Therefore, } \left(\frac{v}{v_c} \right)_0 = \left(\frac{v_i}{v_c} \right)_0 e^{-\frac{t}{\tau}}$$

To solve the equation of the first order of λ , $\frac{d}{dt} \left(\frac{v}{v_c} \right)_1 = -\frac{1}{\tau} \left(\left(\frac{v}{v_c} \right)_1 + \left(\frac{v}{v_c} \right)_0^3 \right)$

$$\text{DSolve} \left[\left\{ \frac{v'[t]}{v_c} = -\frac{1}{\tau} \left(\frac{v[t]}{v_c} + \left(\frac{v_i}{v_c} \right)^3 e^{-\frac{3t}{\tau}} \right), v[0] = 0 \right\}, v, t \right]$$

$$\left\{ \left\{ v \rightarrow \text{Function} \left[\left\{ t \right\}, -\frac{e^{-\frac{3t}{\tau}} (-1 + e^{\frac{2t}{\tau}}) v_i^3}{2 v_c^2} \right] \right\} \right\}$$

$$\text{So, the answer is } \left(\frac{v}{v_c} \right)_1 = \frac{v_i}{v_c} e^{-\frac{t}{\tau}} + \frac{1}{2} \left(e^{-\frac{3t}{\tau}} - e^{-\frac{t}{\tau}} \right) \left(\frac{v_i}{v_c} \right)^3$$

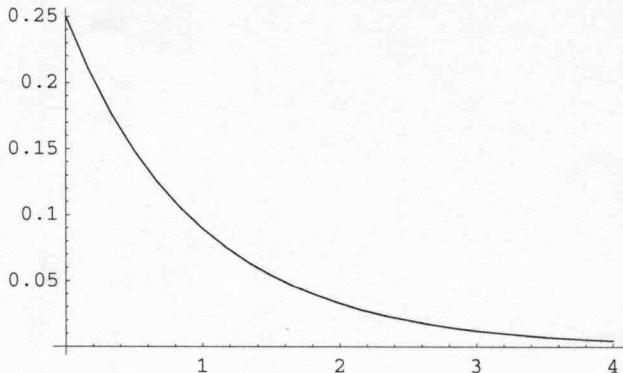
b) the initial velocity is $\frac{v_2}{4}$,

$$\tau = 1; v_i = \frac{v_c}{4}; v_c = 1;$$

$$\text{Vexact} = \text{NDSolve} \left[\left\{ \frac{ve'[t]}{v_c} = -\frac{1}{\tau} \left(\frac{ve[t]}{v_c} + \left(\frac{vi}{v_c} \right)^3 e^{-\frac{3t}{\tau}} \right), ve[0] = vi \right\}, ve, \{t, 0, 4\} \right]$$

$$\left\{ \left\{ ve \rightarrow \text{InterpolatingFunction}[\{0..4.\}], <> \right\} \right\}$$

Plot[Evaluate[ve[t] /. %], {t, 0, 4}]



- Graphics -

$$\text{v0} = \text{NDSolve} \left[\left\{ \text{v0}'[t] = -\frac{1}{\tau} \text{v0}[t], \text{v0}[0] = 1/4 \right\}, \text{v0}, \{t, 0, 4\} \right]$$

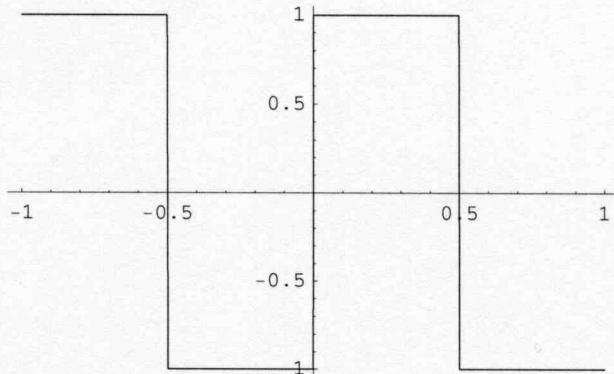
$$\left\{ \left\{ v0 \rightarrow \text{InterpolatingFunction}[\{0..4.\}], <> \right\} \right\}$$

3.(c)

In[1]:= $f = \text{If} [\text{Mod}[t, 1] < 1/2, 1, -1]$

Out[1]= $\text{If} [\text{Mod}[t, 1] < \frac{1}{2}, 1, -1]$

In[2]:= Plot[f, {t, -1, 1}]



Out[2]= - Graphics -

$$\text{In[8]:= } f1 = \sum_{n=1}^1 \left(2 \frac{(1 - (-1)^n)}{n * \pi} \sin[2 * \pi * n * t] \right)$$

$$f3 = \sum_{n=1}^3 \left(2 \frac{(1 - (-1)^n)}{n * \pi} \sin[2 * \pi * n * t] \right)$$

$$f5 = \sum_{n=1}^5 \left(2 \frac{(1 - (-1)^n)}{n * \pi} \sin[2 * \pi * n * t] \right)$$

$$f7 = \sum_{n=1}^7 \left(2 \frac{(1 - (-1)^n)}{n * \pi} \sin[2 * \pi * n * t] \right)$$

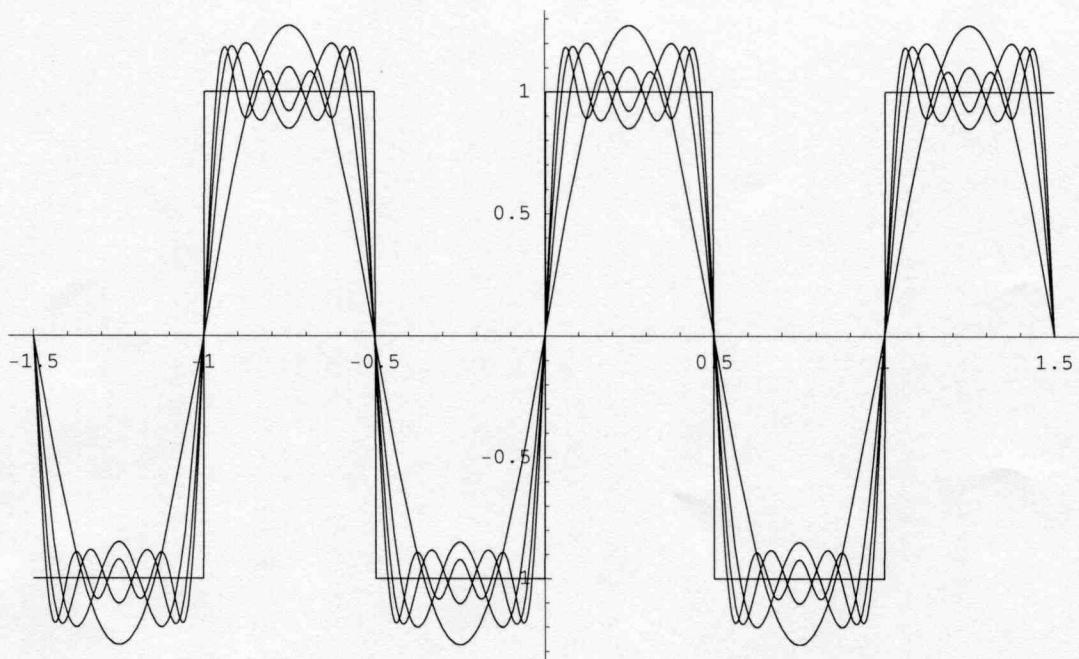
$$\text{Out[8]= } \frac{4 \sin[2 \pi t]}{\pi}$$

$$\text{Out[9]= } \frac{4 \sin[2 \pi t]}{\pi} + \frac{4 \sin[6 \pi t]}{3 \pi}$$

$$\text{Out[10]= } \frac{4 \sin[2 \pi t]}{\pi} + \frac{4 \sin[6 \pi t]}{3 \pi} + \frac{4 \sin[10 \pi t]}{5 \pi}$$

$$\text{Out[11]= } \frac{4 \sin[2 \pi t]}{\pi} + \frac{4 \sin[6 \pi t]}{3 \pi} + \frac{4 \sin[10 \pi t]}{5 \pi} + \frac{4 \sin[14 \pi t]}{7 \pi}$$

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In[12]:= Plot[{f, f1, f3, f5, f7}, {t, -1.5, 1.5}]
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Out[12]= - Graphics -
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3. a) $U(x) = \frac{1}{2}kx^2 + \frac{1}{24}\alpha x^4$ released from at $x=A$ at $t=0$.

at $t=0$ $U(A) = \frac{1}{2}kA^2 + \frac{1}{24}\alpha A^4 = \frac{1}{2}mv^2$ at $x=0$.

$$v^2 = \frac{kA^2}{m} + \frac{\alpha A^4}{12m}$$

$$v = \pm \sqrt{\frac{kA^2}{m} + \frac{\alpha A^4}{12m}}$$

b)

$$F = -\frac{\partial}{\partial x} U(x) = -kx - 6\alpha x^3$$

$$= m\ddot{x}$$

$$\Rightarrow \ddot{x} = -\frac{k}{m}x - \frac{6\alpha}{m}x^3$$

$$x(t) = \left(x_i - \frac{\alpha x_i^3}{192m\omega_0^2} \right) \cos \left[\left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) t \right]$$

$$+ \frac{\alpha x_i^3}{192m\omega_0^2} \cos \left[3 \left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) t \right]$$

$$\dot{x} = - \left(x_i - \frac{\alpha x_i^3}{192m\omega_0^2} \right) \left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) \sin \left[\left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) t \right]$$

$$- \frac{\alpha x_i^3}{192m\omega_0^2} 3 \left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) \sin \left[3 \left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) t \right]$$

when $x(t) = 0$,

if $\left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) t = \frac{\pi}{2}$, then $\begin{cases} \cos \left[\left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) t \right] = 0 \\ \cos \left[3 \left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) t \right] = 0 \end{cases}$ so $x(t) = 0$.

$$\Rightarrow \sin \left[\left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) t \right] = 1 \quad \text{and} \quad \sin \left[3 \left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) t \right] = -1$$

$$\dot{x} = - \left(x_i - \frac{\alpha x_i^3}{192m\omega_0^2} \right) \left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) - \frac{\alpha x_i^3}{192m\omega_0^2} 3 \left(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0} \right) (-1)$$

$$\begin{aligned}
 \ddot{x} &= -x_i \omega_0 - \frac{\alpha x_i^3}{16m\omega_0} + \frac{\alpha x_i^3}{192m\omega_0} + O(x^5) + \frac{\alpha x_i^3}{64m\omega_0^3} + O'(x^5) \\
 &= -x_i \omega_0 + \left(-\frac{1}{16} + \frac{1}{64} + \frac{1}{192} \right) \frac{\alpha x_i^3}{m\omega_0} + O(x^5) \\
 &= -x_i \omega_0 - \frac{12-3-1}{192} \frac{\alpha x_i^3}{m\omega_0} + O(x^5) \\
 &= -x_i \omega_0 - \frac{1}{24} \frac{\alpha x_i^3}{m\omega_0} + \dots
 \end{aligned}$$

(C) from a) $\omega = \dot{x} = \pm \sqrt{\frac{kA^2}{m} + \frac{\alpha A^4}{12m}}$

$$\begin{aligned}
 \dot{x} &= \pm \sqrt{\frac{k}{m}} A \sqrt{1 + \frac{\alpha A^2}{12k}} \quad \text{where } \omega_0 = \sqrt{\frac{k}{m}}. \\
 &\approx \pm \omega_0 A \left(1 + \frac{1}{2} \cdot \frac{\alpha A^2}{12k} \right) + \dots \\
 &\approx \underbrace{\pm \left(\omega_0 A + \frac{\alpha A^3}{24k} + \dots \right)}_{\text{as above.}}
 \end{aligned}$$

Note

At part (b), I choose $(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0})t = \frac{\pi}{2}$,

$$\text{then, } \dot{x} = -A \omega_0 - \frac{\alpha A^3}{24m\omega_0},$$

If I choose $(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0})t = -\frac{\pi}{2}$,

$$\text{then } \sin \left[(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0})t \right] = -1, \quad \sin \left[3(\omega_0 + \frac{\alpha x_i^2}{16m\omega_0})t \right] = 1$$

So, we can have

$$\dot{x} = \underbrace{x_i \omega_0 + \frac{\alpha x_i^3}{24m\omega_0} + \dots}_{\text{,}}$$

3. d) A must be small,

$$\frac{\alpha A^2}{24k} \ll 1$$

$$A^2 \ll \frac{k}{\alpha}$$

$$A \ll \sqrt{\frac{k}{\alpha}}$$

4. a) $U(x) = \frac{1}{2}kx^2 + \frac{1}{24}\alpha x^4$; rest at $x=A$ at $t=0$

by the energy conservation.

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 + \frac{1}{24}\alpha x^4 = \frac{1}{2}kA^2 + \frac{1}{24}\alpha A^4 = E_{\text{tot}}$$

$$\ddot{x}^2 = \frac{k}{m}(A^2 - x^2) + \frac{2\alpha}{12m}(A^4 - x^4)$$

$$\frac{dx}{dt} = \sqrt{\frac{k}{m}(A^2 - x^2) + \frac{\alpha}{12m}(A^4 - x^4)}$$

from $-A$ to A . it takes a half of the period, T .

$$\int_0^{T/2} dt = \int_{-A}^A \frac{dx}{\sqrt{\frac{k}{m}(A^2 - x^2) + \frac{\alpha}{12m}(A^4 - x^4)}}$$

$$T = 2 \int_{-A}^A \frac{dx}{\sqrt{\left(\frac{k}{m}(A^2 - x^2) + \frac{\alpha}{12m}(A^4 - x^4)\right)}} //$$

b)

$$2 \int_{-A}^A \frac{d\left(\frac{x}{A}\right) \cdot A}{\sqrt{\frac{k}{m}\left(1 - \left(\frac{x}{A}\right)^2\right) + \frac{\alpha A^2}{12m}\left(1 - \left(\frac{x}{A}\right)^4\right)}} = 2 \cdot 2 \int_0^1 \frac{dy}{\sqrt{\frac{k}{m}\left(1 - y^2\right) + \frac{\alpha A^2}{12m}\left(1 - y^4\right)}}$$

$$y = \frac{x}{A}$$

$$\begin{aligned}
 &= 4 \int_0^1 \frac{1}{\sqrt{\frac{k}{m}(1-y^2)}} \frac{dy}{\sqrt{1 + \frac{\alpha A^2}{12k} (1+y^2)}} \\
 &\approx 4 \int_0^1 \frac{1}{\sqrt{\frac{k}{m}(1-y^2)}} \left(1 - \frac{1}{2} \frac{\alpha A^2}{12k} (1+y^2) + \dots \right) dy \\
 &\approx 4 \int_0^1 \frac{dy}{\sqrt{\frac{k}{m}(1-y^2)}} - 4 \int_0^1 \frac{\alpha A^2}{24k} \frac{1+y^2}{\sqrt{\frac{k}{m}(1-y^2)}} dy \\
 &= \cancel{4} \cdot \sqrt{\frac{m}{k}} \frac{\pi}{2} - \cancel{4} \frac{\alpha A^2}{24k} \sqrt{\frac{m}{k}} \frac{3\pi}{4} = 2\pi \sqrt{\frac{m}{k}} \left(1 - \frac{\alpha A^2}{16k} \right) \\
 &= \frac{2\pi}{\omega_0} \left(1 - \frac{\alpha A^2}{16k} \right) \\
 &\therefore T = \underline{\frac{2\pi}{\omega_0} \left(1 - \frac{\alpha A^2}{16k} \right)} \quad //
 \end{aligned}$$

c)

$$\begin{aligned}
 \omega &= \omega_0 \left(1 + \frac{\alpha A^2}{16m\omega_0^2} \right) \\
 \gamma &= \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0 \left(1 + \frac{\alpha A^2}{16m\omega_0^2} \right)} \approx \frac{2\pi}{\omega_0} \left(1 - \frac{\alpha A^2}{16m\omega_0^2} \right) \\
 &\left(\omega_0^2 = \frac{k}{m} \right) \\
 \gamma &\approx \frac{2\pi}{\omega_0} \left(1 - \frac{\alpha A^2}{16k} \right) \\
 &\underline{\hspace{10em}} \quad //
 \end{aligned}$$