Homework 3

1) Express the function $f(t)=A\cos^3(\omega t)$ in the form of a Fourier series $f(t)=\sum_{n=-\infty}^{\infty} c_n e^{-i n\omega t}$. Explain why the series truncates, *i.e.* why c_n =0 for $|n|$ >3.

2) This second problem is to get a feel for Fourier series. In class we argued that they were useful for smooth periodic functions. Here I want you to show that they are even useful to describe discontinuous functions where the approximation can do well except at the point of discontinuity. Consider the square wave:
$f(t)=\begin{cases} +1 \ for \ 0 \le Mod_1(t) < \frac{1}{2} \\ -1 \ for \ \frac{1}{2} \le Mod_1(t) < 1 \end{cases}$ where $Mod_1(t)$ is the fractional part of *t*, *i.e.*
$Mod_1(t)=t$ for $0 \le t < 1$, $Mod_1(t)=t-1$ for $1 \le t < 2$, $Mod_1(t)=t-2$ for $2 \le t < 3$, and so forth. Thus *f(t)* alternates between positive and negative one in a periodic manner with a period of unity.

   a) Assume *f(t)* can be written as a Fourier series of the form $f(t)=\sum_{n=-\infty}^{\infty} c_n e^{-i2\pi nt}$ and show that the c_n coefficients $c_n=\begin{cases} \frac{-2i}{n\pi} \ for \ n \ odd \\ 0 \ for \ n \ even \end{cases}$.

   b) Since *f(t)* is real we should have $c_{-n}=c_n^*$ verify that this true. Explain the significance of the fact the c_n coefficients are pure imaginary.

   c) Plot the exact expression and 4 approximations based on the Fourier series a truncation for n<2, n<4, n<6 and n<8. Comment on the apparent converngence. Where does the expansion do poorly and why?

3) Consider a particle of mass *m* moving in the following anharmonic potential:
$U(x)=\frac{1}{2}kx^2+\frac{1}{24}\alpha x^4$. The particle is released from rest at *x=A* at *t=0*.

   a) Use energy conservation to determine the speed of the particle when it reaches *x=0*.

   b) Use the approximate expression based on Fourier analysis to determine the velocity at *x=0*. You should work at lowest nontrivial order (including the 3ω term in the expansion)

   c) Show that the two expressions derived in problem 3a) and 3b) are consistent, *i.e.* they differ only beyond the order calculated in the approximation

d) Specify the condition on A for which you expect your answer in b) to be valid.

4) Consider a particle of mass m moving in the same anharmonic potential as in problem 3): $U(x) = \frac{1}{2} k x^2 + \frac{1}{24} \alpha x^4$. The particle is released from rest at $x=A$ at $t=0$.

a) Derive an expression for the period of this system as a function of the amplitude using the energy conservation method described in class. The expression can be left in the form of an integral. Hint: how is the period related to the time it takes to go from $-A$ to A where A is the amplitude?

b) Derive an approximate expression for the period as a function of the amplitude starting from the exact expression in part a). Your zeroth order result should give the period for the harmonic oscillator and corrections should grow with A. Hint: write the integral in terms of dimensionless variables.

e) Show that this answer is equivalent to the one found in class using Fourier methods, *i.e.* they differ only beyond the order calculated in the approximation