PHYS 374

Homework #2

Solution

1) a. $F(v) = -\alpha v e^{\beta v^2}$

- the exponent βv^2, is dimensionless.

$\beta v^2 \sim 1$ $\qquad \underline{v_c \sim \beta^{-1/2}}$

- $F = m\dfrac{dv}{dt} = -\alpha v e^{\beta v^2}$

$\dfrac{dv}{dt} = [LT^{-2}] \qquad v = [LT^{-1}]$

$\dfrac{v}{\frac{dv}{dt}} = [T] \sim t \longrightarrow \underline{t = \dfrac{m}{\alpha}}$

b. $F(v) = m\dfrac{dv}{dt} = -\alpha v e^{\beta v^2}$

$\beta = \dfrac{1}{v_c^2}$

$\dfrac{dv}{dt} = -\dfrac{\alpha}{m} v e^{\left(\frac{v}{v_c}\right)^2} \xrightarrow[\text{devide by } v_c]{} \underline{\dfrac{d\left(\frac{v}{v_c}\right)}{dt} = -\dfrac{\alpha}{m}\dfrac{v}{v_c} e^{\left(\frac{v}{v_c}\right)^2}}$

2) a.

$$\left(\frac{v}{v_c}\right) = \left(\frac{v}{v_c}\right)_0 + \lambda\left(\frac{v}{v_c}\right)_1 + \lambda^2\left(\frac{v}{v_c}\right)_2 + \cdots$$

$$\left\{\left(\frac{v}{v_c}\right)\right\}^2 = \left(\frac{v}{v_c}\right)_0^2 + 2\lambda\left(\frac{v}{v_c}\right)_0\left(\frac{v}{v_c}\right)_1 + \lambda^2\left\{\left(\frac{v}{v_c}\right)_1\right\}^2 + \lambda^2\left(\frac{v}{v_c}\right)_0\left(\frac{v}{v_c}\right)_2 + \cdots$$

$$\exp\left[\lambda\left(\frac{v}{v_c}\right)^2\right] = \exp\left[\lambda\left(\frac{v}{v_c}\right)_0^2 + 2\lambda^2\left(\frac{v}{v_c}\right)_0\left(\frac{v}{v_c}\right)_1 + \cdots\right]$$

$$e^x \simeq 1 + x + \frac{1}{2}x^2 + \cdots$$

$$\exp\left[\lambda\left(\frac{v}{v_c}\right)_0^2 + 2\lambda^2\left(\frac{v}{v_c}\right)_0\left(\frac{v}{v_c}\right)_1 + \cdots\right]$$

$$= 1 + \left\{\lambda\left(\frac{v}{v_c}\right)_0^2 + 2\lambda^2\left(\frac{v}{v_c}\right)_0\left(\frac{v}{v_c}\right)_1 + \cdots\right\} + \frac{1}{2}\left\{\lambda\left(\frac{v}{v_c}\right)_0^2 + 2\lambda^2\left(\frac{v}{v_c}\right)_0\left(\frac{v}{v_c}\right)_1 + \cdots\right\}^2 + \cdots$$

$$\cong 1 + \lambda\left(\frac{v}{v_c}\right)_0^2 + \lambda^2\left\{\frac{1}{2}\left(\frac{v}{v_c}\right)_0^4 + 2\left(\frac{v}{v_c}\right)_0\left(\frac{v}{v_c}\right)_1\right\} + \cdots$$

b)

$$\frac{d\left(\frac{v}{v_c}\right)}{dt} = -\frac{1}{\tau}\left(\frac{v}{v_c}\right)\exp\left[\left(\frac{v}{v_c}\right)^2\right]$$

$$\left(\frac{v}{v_c}\right) = \left(\frac{v}{v_c}\right)_0 + \lambda\left(\frac{v}{v_c}\right)_1 + \lambda^2\left(\frac{v}{v_c}\right)_2 + \cdots$$

$$\frac{d\left(\frac{v}{v_c}\right)}{dt} = \frac{d\left(\frac{v}{v_c}\right)_0}{dt} + \lambda\frac{d\left(\frac{v}{v_c}\right)_1}{dt} + \lambda^2\frac{d\left(\frac{v}{v_c}\right)_2}{dt} + \cdots$$

$$= -\frac{1}{\tau}\left(\frac{v}{v_c}\right)\exp\left[\left(\frac{v}{v_c}\right)^2\right]$$

$$= -\frac{1}{\tau}\left[\left(\frac{v}{v_c}\right)_0 + \lambda\left(\frac{v}{v_c}\right)_1 + \cdots\right]\left[1 + \lambda\left(\frac{v}{v_c}\right)_0^2 + \cdots\right]$$

$$= -\frac{1}{\tau}\left[\left(\frac{v}{v_c}\right)_0 + \lambda\left(\frac{v}{v_c}\right)_0\left(\frac{v}{v_c}\right)_0^2 + \lambda\left(\frac{v}{v_c}\right)_1 + \cdots\right]$$

Comparing each order of λ,

$$\lambda^0 : \quad \frac{d\left(\frac{v}{v_c}\right)}{dt} = -\frac{1}{\tau}\left(\frac{v}{v_c}\right)_0$$

$$\lambda^1 : \quad \frac{d\left(\frac{v}{v_c}\right)}{dt} = -\frac{1}{\tau}\left[\left(\frac{v}{v_c}\right)_0^3 + \left(\frac{v}{v_c}\right)_1\right]$$

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3. a) $\frac{d}{dt}\left(\frac{v}{v_c}\right)_0 = -\frac{1}{\tau}\left(\frac{v}{v_c}\right)_0$ with initial velocity $\left(\frac{v}{v_c}\right)_0$.

$$\frac{1}{\left(\frac{v}{v_c}\right)_0}\frac{d}{dt}\left(\frac{v}{v_c}\right)_0 = -\frac{1}{\tau} \qquad \int_{v_i}^{v}\frac{1}{\left(\frac{v}{v_c}\right)_0}\,d\left(\frac{v}{v_c}\right)_0 = -\int_0^t \frac{1}{\tau}\,dt$$

$$\text{Log}\left[\frac{v}{v_c}\right] - \text{Log}\left[\frac{v_i}{v_c}\right] = -\frac{t}{\tau} \qquad \left(\frac{v}{v_c}\right)_0 \Big/ \left(\frac{v_i}{v_c}\right)_0 = e^{-\frac{t}{\tau}}$$

When t = 0, $\left(\frac{v}{v_c}\right)_0 \Big/ \left(\frac{v_i}{v_c}\right)_0 = 1 = \left(\frac{v}{v_c}\right)_0$ Therefore, $\left(\frac{v}{v_c}\right)_0 = \left(\frac{v_i}{v_c}\right)_0 e^{-\frac{t}{\tau}}$

To solve the equation of the first ordr of λ, $\frac{d}{dt}\left(\frac{v}{v_c}\right)_1 = -\frac{1}{\tau}\left(\left(\frac{v}{v_c}\right)_1 + \left(\frac{v}{v_c}\right)_0^3\right)$

$$\textbf{DSolve}\left[\left\{\frac{v'[t]}{vc} == -\frac{1}{\tau}\left(\frac{v[t]}{vc} + \left(\frac{vi}{vc}\right)^3 e^{-\frac{3t}{\tau}}\right),\ v[0] == 0\right\},\ v,\ t\right]$$

$$\left\{\left\{v \to \left(\frac{e^{-\frac{3\#1}{\tau}}\left(vi^3 - e^{\frac{2\#1}{\tau}} vi^3\right)}{2\,vc^2}\ \&\right)\right\}\right\}$$

So, the answer is $\left(\frac{v}{v_c}\right)_1 = \frac{v_i}{v_c} e^{-\frac{t}{\tau}} + \frac{1}{2}\left(e^{-\frac{3t}{\tau}} - e^{-\frac{t}{\tau}}\right)\left(\frac{v_i}{v_c}\right)^3$

b) the initial velocity is $\frac{v_2}{4}$,

$$\tau = 1;\ vi = \frac{vc}{4};\ vc = 1;$$

$$\textbf{Vexact} = \textbf{NDSolve}\left[\left\{\frac{ve'[t]}{vc} == -\frac{1}{\tau}\left(\frac{ve[t]}{vc} + \left(\frac{vi}{vc}\right)^3 e^{-\frac{3t}{\tau}}\right),\ ve[0] == vi\right\},\ ve,\ \{t, 0, 4\}\right]$$

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{{ve → InterpolatingFunction[{{0., 4.}}, <>]}}
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$$\textbf{V0} = \textbf{NDSolve}\left[\left\{v0'[t] == -\frac{1}{\tau}\,v0[t],\ v0[0] == 1/4\right\},\ v0,\ \{t, 0, 4\}\right]$$

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{{v0 → InterpolatingFunction[{{0., 4.}}, <>]}}
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$$\textbf{V1} = \textbf{NDSolve}\left[\left\{v1'[t] == -\frac{1}{\tau}\left(v1[t] + \left(\frac{1}{4}e^{-\frac{t}{\tau}}\right)^3\right),\ v1[0] == 1/4\right\},\ v1,\ \{t, 0, 4\}\right]$$

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{{v1 → InterpolatingFunction[{{0., 4.}}, <>]}}
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Plot[ {Evaluate[v0[t] /. V0],
  Evaluate[v1[t] /. V1], Evaluate[ve[t] /. Vexact]}, {t, 0, 4}]
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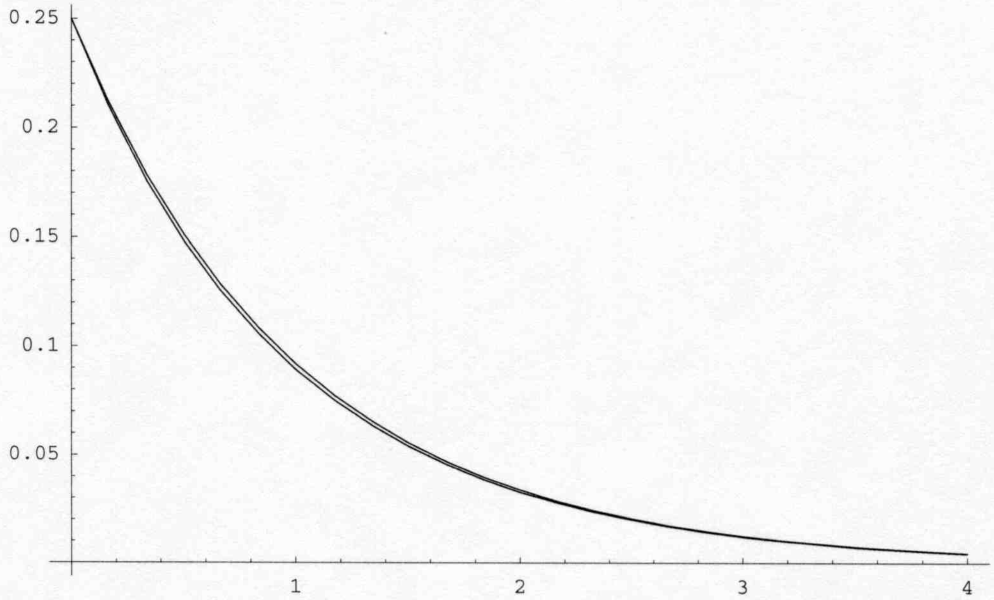

- Graphics -

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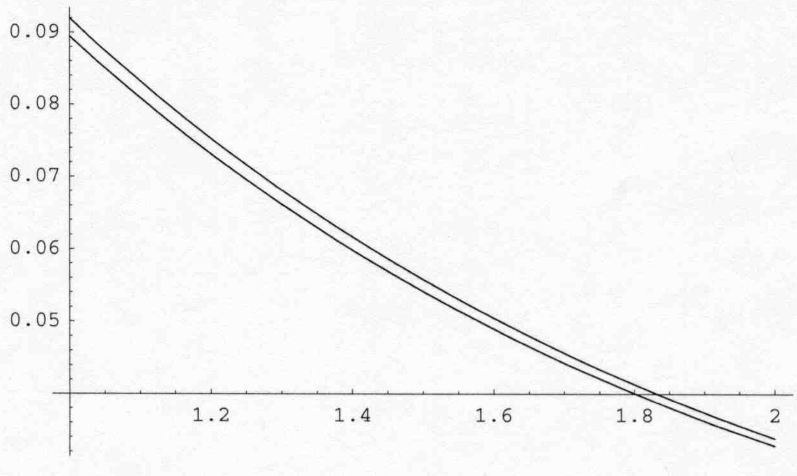
Plot[ {Evaluate[v0[t] /. V0], Evaluate[v1[t] /. V1]}, {t, 1, 2}]
```



- Graphics -

4. a) $U(x) = U_0 (1 - e^{-(\frac{x}{l})^2})$

$U'(x) = U_0 \frac{2x}{l^2} e^{-(\frac{x}{l})^2}$

$U''(x) = U_0 \left(\frac{2}{l^2} e^{-(\frac{x}{l})^2} + \frac{2x}{l^2}\left(-\frac{2x}{l^2}\right) e^{-(\frac{x}{l})^2} \right) = U_0 \left(\frac{2}{l^2} e^{-(\frac{x}{l})^2} - \left(\frac{2x}{l^2}\right)^2 e^{-(\frac{x}{l})^2} \right)$

$U''(0) = \frac{U_0}{l^2} = k$ assume $x_0 = 0$

$\therefore \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2U_0}{ml^2}}$

(b) $U(x) = U_0 \left[\cos\left(\frac{x}{l}\right) + \cos\left(\frac{2x}{l}\right) \right]$

$U'(x) = U_0 \left[-\frac{1}{l}\sin\left(\frac{x}{l}\right) - \frac{2}{l}\sin\left(\frac{2x}{l}\right) \right]$

$U''(x) = U_0 \left[-\frac{1}{l^2}\cos\left(\frac{x}{l}\right) - \frac{4}{l^2}\cos\left(\frac{2x}{l}\right) \right]$

find the minimum of $U(x)$. $\therefore$ $U'(x_0) = U_0 \left(-\frac{\sin(\frac{x_0}{l})}{l} - \frac{2\sin(\frac{2x_0}{l})}{l} \right) = 0$

$\sin\frac{x_0}{l} = -2\sin\left(\frac{2x_0}{l}\right) = -4\sin\left(\frac{x_0}{l}\right)\cos\left(\frac{x_0}{l}\right)$

$\cos\frac{x_0}{l} = -\frac{1}{4}$ $\cos\left(\frac{2x_0}{l}\right) = \cos^2\frac{x_0}{l} - \sin^2\frac{x_0}{l} = 2\cos^2\frac{x_0}{l} - 1 = \frac{2}{16} - 1 = -\frac{7}{8}$

So, $U''(x_0) = -\frac{U_0}{l^2}\left(\cos\frac{x_0}{l} + 4\cos\frac{2x}{l}\right) = +\frac{U_0}{l^2}\left(+\frac{1}{4} + 4\cdot\left(+\frac{7}{8}\right)\right) = \frac{15U_0}{4l^2}$

$\therefore \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{15U_0}{4ml^2}}$

(c) $U(x) = U_0 \sin\frac{x}{l}$

$U'(x) = \frac{U_0}{l}\cos\frac{x}{l}$ $U''(x) = -\frac{U_0}{l}\sin\left(\frac{x}{l}\right)$

$U'(x_0) = \frac{U_0}{l}\cos\frac{x_0}{l} = 0$ $x_0 = \frac{3l}{2}\pi$ (minimum) $\frac{l}{2}\pi$; maximum X

$U''(x_0) = -\frac{U_0}{l}\sin\left(\frac{3}{2}\cancel{l}\pi\frac{1}{\cancel{l}}\right) = \frac{U_0}{l}$ $\therefore \omega = \sqrt{\frac{U_0}{ml}}$

5. a) initially at rest at position A $\rightarrow \dot{x}(0) = 0 \quad x(0) = A$

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2$$

$$\dot{x}^2 = \frac{k}{m}(A^2 - x^2) = w_o^2 (A^2 - x^2)$$

$$\frac{dx}{dt} = \pm\sqrt{w_o^2 (A^2 - x^2)}$$

Consider only positive, $\displaystyle\int \frac{dx}{\sqrt{A^2 - x^2}} = \pm\int w_o \, dt$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}$$

$$\rightarrow \int \frac{dx}{\sqrt{A^2 - x^2}} = \pm\int w_o \, dt$$

$$\arcsin\left(\frac{x}{A}\right) = \pm w_o t + C$$

$$\frac{x}{A} = \sin(\pm w_o t + C)$$

Boundary condition

$x = A \sin(\pm w_o t + C) \qquad x(0) = A \sin C = A \qquad C = \frac{\pi}{2}$

$x = A \cos w_o t \qquad$ where $\quad w_o^2 = \frac{k}{m}$

check! $\dot{x} = -w_o A \sin w_o t \qquad \dot{x}(0) = 0$

b) initially at $x=0$ and moving velocity v. $(v>0)$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv^2$$

$$\dot{x}^2 = v^2 - \frac{k}{m}x^2 = v^2 - w_0^2 x^2 \qquad \text{where } w_0^2 = \frac{k}{m}$$

$$\frac{dx}{dt} = \pm\left(v^2 - w_0^2 x^2\right)^{1/2} = \pm w_0\sqrt{\left(\frac{v}{w_0}\right)^2 - x^2}$$

$$\int \frac{dx}{\sqrt{\left(\frac{v}{w_0}\right)^2 - x^2}} = \pm\int w_0\, dt.$$

Similarly, use the formula. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$

$$\arcsin\frac{x}{\frac{v}{w_0}} = \pm w_0 t + C$$

$$\frac{w_0}{v}x = \sin(C \pm w_0 t)$$

$$x = \frac{v}{w_0}\sin(C \pm w_0 t)$$

~~B~~ Boundary Conditions. $x(0)=0$, $\dot{x}(0)=v$

$$x(0) = \frac{v}{w_0}\sin C = 0 \quad \rightarrow C = 0$$

$$\dot{x}(0) = \pm\frac{v}{w_0}w_0\cos(\pm w_0 \cdot 0) = \pm\frac{v}{\cancel{w_0}}\cancel{w_0} = \pm v, \quad \rightarrow \text{we need to choose } +.$$

$$\therefore\ x(t) = \frac{v}{w_0}\sin(w_0 t) \qquad \text{where } w_0^2 = \frac{k}{m}$$