Physics 374

1) In class we studied the problem of a falling body under the influence of gravity and air resistance exponentially approaching its terminal velocity. In this problem we will study possible corrections to this result due to nonlinearity. For simplicity , we will consider the case where gravity does not play a role so that the terminal velocity is zero. We will consider a force of the form $F(v) = -\alpha v e^{\beta v^2}$.
   a) Use dimensional analysis to find the characteristic time τ and charactertistic velocity scale v_c .
   b) Show the differential equation for the motion may be written as
   $$\frac{d\left(\frac{v}{v_c}\right)}{dt} = \frac{-1}{\tau}\left(\frac{v}{v_c}\right)\exp\left(\left(\frac{v}{v_c}\right)^2\right).$$

2) As we are interested in the behavior as the velocity asymptotes to zero we are effectively doing a low velocity expansion. Thus we expect the term in the exponential to be small and can use it as an expansion parameter. Thus we can rewrite the preceding equation as $\frac{d\left(\frac{v}{v_c}\right)}{dt} = \frac{-1}{\tau}\left(\frac{v}{v_c}\right)\exp\left(\lambda\left(\frac{v}{v_c}\right)^2\right)$ and write our velocity as a power series in λ: $\left(\frac{v}{v_c}\right) = \left(\frac{v}{v_c}\right)_0 + \lambda\left(\frac{v}{v_c}\right)_1 + \lambda^2\left(\frac{v}{v_c}\right)_2 + \lambda^3\left(\frac{v}{v_c}\right)_3 + \ldots.$

   a) In order to proceed we need to expand out the exponential as a series in λ. Show that up to order λ^2 ,
   $$\exp\left(\lambda\left(\frac{v}{v_c}\right)^2\right) = 1 + \lambda\left(\frac{v}{v_c}\right)_0^2 + \lambda^2\left(\frac{1}{2}\left(\frac{v}{v_c}\right)_0^4 + 2\left(\frac{v}{v_c}\right)_0\left(\frac{v}{v_c}\right)_1\right) + \ldots$$
   b) Show that the equations for first two terms in the velocity expansion are $\frac{d}{dt}\left(\frac{v}{v_c}\right)_0 = \frac{-1}{\tau}\left(\frac{v}{v_c}\right)_0$ $\qquad$ $\frac{d}{dt}\left(\frac{v}{v_c}\right)_1 = \frac{-1}{\tau}\left(\left(\frac{v}{v_c}\right)_1 + \left(\frac{v}{v_c}\right)_0^3\right)$.
   The second term is a correction to the simple exponential.

3) To solve these equations we need initial conditions. As noted in class the initial conditions can be chosen so that the initial velocity is entirely contained in $\left(\frac{v}{v_c}\right)_0$.

a) Solve the equations in 2b) using these boundary conditions. (The first dif. eq. will just give an exponential of the ususal sort. To solve the second use techniques from your dif. eq. class or alternatively turn to Mathematica's Dsolve .

b) For the case where the initial velocity is $\frac{v_2}{4}$, solve the full differential equation numerically (there is no analytic solution---use Mathematica's NDSolve or some other numerical package) . Plot the numerical solution along with the lowest order approximate expression and approximate expression including the correction term. Do your result make sense.

4) Consider a conservative one-dimensional system consisting of a particle of mass, *m*, moving in a potential. For the following potentials, determine the frequency of small amplitude oscillations near the minimum. In all these problems you may take *l* and U_0 to be positive. (Hint: the minimum need not be at x=0)

a) $U(x)=U_0\left(1-e^{-(x/l)^2}\right)$
b) $U(x)=U_0\left[\cos(x/l)+\cos(2x/l)\right]$
c) $U(x)=U_0\left[\sin(x/l)\right]$

5) Consider a particle of mass *m* in a harmonic potential around the origin characterized by a spring constant *k*. In this problem I want you to use the integral expression based on energy conservation (quadratures) to solve for the motion. The integral you get can be done analytically---derive it yourself, look it up on a table or use Mathematica---and *t(x)* can easily be inverted to get *x(t).*

a) Find the motion if the particle is initially at rest at position *A.*
b) Find the motion if the particle is initially at *x=0* and moving with velocity *v* (with *v>0*).