PHYS 374
Home work 1

Problem 1

The differential equation we're working with is $\frac{dv(t)}{dt} = \frac{\lambda}{\tau}(v_t - v(t))$, with v_t andλ constant. We wish to find a Taylor expansion for $v(t)$ that will be nice for small times. Let $v = v_0 + \lambda v_1 + \lambda^2 v_2 + \ldots$ up to second order. We can simply plug this into the differential equation:

$$v' = v_0' + \lambda v_1' + \lambda^2 v_2' = \frac{\lambda}{\tau}(v_t - v_0 - \lambda v_1 - \lambda^2 v_2 - \ldots)$$

Then we group terms on the left by order in λ

$$v_0' + \lambda v_1' + \lambda^2 v_2' = \frac{\lambda}{\tau}(v_t - v_0) - \frac{\lambda^2}{\tau}v_1 - \frac{\lambda^3}{\tau}v_2 - \ldots$$

and since we're working only to second order in λ, we can toss out the λ^3 term. The terms on the left and right must match up by λ, so we have (setting $\lambda = 1$)

$$v_0' = 0 \qquad v_1' = +\frac{(v_t - v_0)}{\tau} \qquad v_2' = -\frac{v_1}{\tau}$$

All of these are easy to solve. By integration $v_0 = k$, with k a constant, and since at small times the velocity ought to be close to the initial velocity, k should be v_i. So $\boxed{v_0 = v_i}$. We can plug this into the DE for v_1, and integrate, getting $\boxed{v_1 = +\frac{(v_t - v_i)t}{\tau}}$, where we're leaving out the constant of integration, since it must be zero as we've already used up our initial condition. Next, we plug our result into the last remaining DE, and integrate to obtain $\boxed{v_2 = \frac{-(v_t - v_i)t^2}{2\tau^2}}$ where we again left out the constant of integration for the same reason.

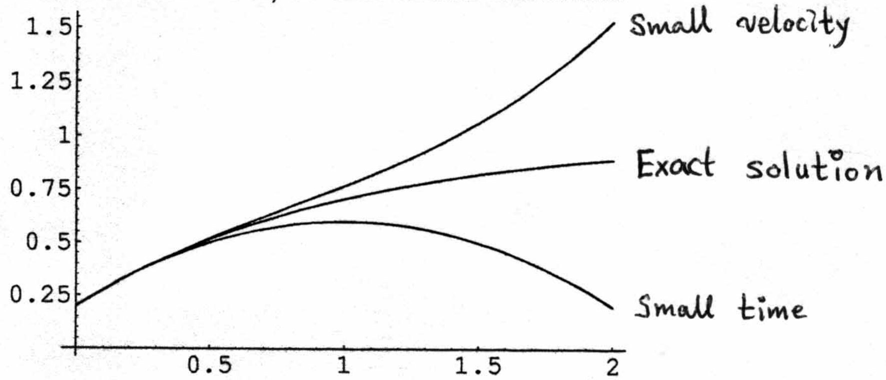
The resulting series for $v(t)$ up to terms of second-order is

$$v(t) = v_i + \frac{(v_t - v_i)t}{\tau} - \frac{(v_t - v_i)t^2}{2\tau^2} + \ldots \tag{1}$$

Problem 2

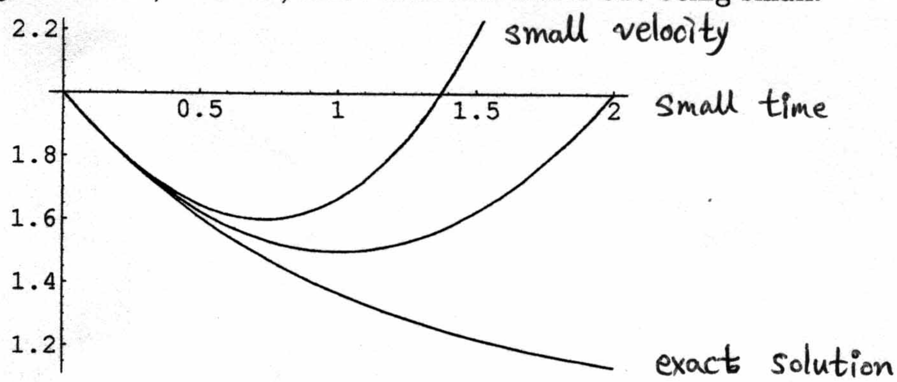
We have two expansions for $v(t)$, the one derived above, for small velocities, and the one derived in class for small velocities. We can plot these two expansions, up to second order, on a graph with the exact solution.

First, we set $v_t = \frac{v_i}{5}$ (as well as $\tau = 1, v_t = 1$, to make Mathematica happy - but this is just a choice of units). Since here the initial velocity is small compared to the terminal velocity, we would expect our small time expansion to work nicely. Also, the small-time expansion should work, as time starts out small.



Both expansions work quite nicely, though the small-velocity expansions to stick closer to the exact solution for longer. This is probably because elapsed time gets to τ faster than the velocity approaches v_t, and we shouldn't generally expect a small-time approximation to work well when time gets to be on the order of τ.

Next, we set $v_i = 2v_t$, and do the same plot. Now, the initial velocity is big, compared to v_t, and the small-velocity ought to crash and burn in short order. The small-time expansion ought to work, however, since time still starts out being small.



Clearly, the small-velocity indeed sucks for these initial conditions. The small-time expansion does a bit better, but doesn't seem particularly spectacular, either. It does perform nicely as long as the time is small compared to τ (which of course is all we can generally expect).

Problem 3

Part a

The problem is to come up with an equation modeling an object falling through a medium with a resistance that goes like the square of the velocity. There are two forces on such on object, the force of gravity and the resistance force from the medium. If we call the direction of the gravity force positive, the two forces look like mg and $-\beta v^2$, respectively, where m is the mass of the object, g is the acceleration of gravity, β is a constant that controls the strength of the resistance force. By Newton's law, $F = mv'$, so our model is $\boxed{mv' = mg - \beta v^2}$.

Part b

The resistance force, being a force, has units of kg meters/sec^2, and v^2 has units of $meters/sec^2$. Therefore β has to have units of $kg/meters$. Next, we want to find combinations of m, g, β that have units of velocity and time. At the terminal velocity, the force of gravity and the resistance force must balance, so

$$mg = \beta v_t^2 \Rightarrow \sqrt{\frac{mg}{\beta}} = v_t$$

Next, g has units of meters/sec^2, so since v_t has units of meters/sec $\boxed{\tau = v_t/g} = \sqrt{\frac{m}{g\beta}}$.

Part c

We start with

$$mv' = mg - \beta v^2$$

Dividing both sides by m and v_t, we get

$$\left(\frac{v}{v_t}\right)' = \frac{g}{v_t} - \frac{\beta}{m}\frac{v^2}{v_t}$$

Next, it's clear that $g/v_t = \frac{1}{\tau}$, and

$$\frac{\beta}{m}\frac{v^2}{v_t} = \frac{\beta v_t}{m}\left(\frac{v}{v_t}\right)^2$$

Now, what's $\frac{\beta v_t}{m}$? Well,

$$\frac{\beta v_t}{m} = \frac{\beta\sqrt{\frac{gm}{\beta}}}{m} = \sqrt{\frac{\beta^2}{m^2}\frac{gm}{\beta}} = \sqrt{\frac{g\beta}{m}} = \frac{1}{\tau}$$

So finally we have

$$\left(\frac{v}{v_t}\right)' = \frac{1}{\tau}\left(1 - \left(\frac{v}{v_t}\right)^2\right)$$

which is what we needed to show.

Problem 4

Part a

Here the differential equation we're dealing with is

$$\left(\frac{v}{v_t}\right)' = \frac{1}{\tau}\left(1 - \lambda\left(\frac{v}{v_t}\right)^2\right)$$

Out of sloth, let's just rename $\frac{v}{v_t}$ to v (and propagate this change into all the v_0's and whatnots), and just make sure to remember to change back later. What we're looking for is an expansion for v that's good for low velocities. So we're looking for the coefficients in an equation that looks like this:

$$v = v_0 + \lambda v_1 + \lambda^2 v_2 + \ldots$$

Now, we can differentiate both sides of the above and plug in our known equation for v' like this:

$$v' = v_0' + \lambda v_1' + \lambda^2 v_2' + \ldots \qquad \frac{1}{\tau}(1 - v^2) = v_0' + \lambda v_1' + \lambda^2 v_2' + \ldots$$

Let's figure out what v^2 is:

$$v^2 = (v_0 + \lambda v_1 + \lambda^2 v_2)^2 = v_0^2 + \lambda(2v_0 v_1) + \lambda^2 v_1^2 + \ldots$$

(where anything of greater than second order got thrown out). Next we can slip this expression into our DE: we can expand v^2 and collect terms according to powers of λ

$$\frac{1}{\tau} - \lambda\frac{v^2}{\tau} = \frac{1}{\tau} - \lambda\left(\frac{v_0^2}{\tau} + \frac{\lambda 2 v_0 v_1}{\tau} + \frac{\lambda^2 v_0 v_2}{\tau} + \frac{\lambda^2 v_1}{\tau} + \ldots\right) = v_0' + \lambda v_1' + \lambda^2 v_2' + \ldots$$

$$0 = (v_0' - \frac{1}{\tau}) + \lambda(v_1' + \frac{v_0^2}{\tau}) + \lambda^2(v_1' + 2\frac{v_0 v_1}{\tau}) + \ldots$$

and then since the left be zero term-by-term, we've found some differential equations all the coefficients must satify. Namely,

$$\left(\frac{v}{v_t}\right)_0' = \frac{1}{\tau}$$

$$\left(\frac{v}{v_t}\right)_1' = -\frac{1}{\tau}\left(\frac{v}{v_t}\right)_0^2$$
$$\left(\frac{v}{v_t}\right)_2' = -\frac{2}{\tau}\left(\frac{v}{v_t}\right)_0\left(\frac{v}{v_t}\right)_1$$
where we've renamed all the v's back to their original designations.

Part b

We now want to solve the differential equations in part a, imposing the below boundary conditions at $t = 0$.
$$\left(\frac{v}{v_t}\right)_0 = \left(\frac{v_i}{v_t}\right) \qquad \left(\frac{v}{v_t}\right)_1 = \left(\frac{v_i}{v_t}\right)_2 = 0$$
These boundary conditions come from the idea that we've looking for a perturbative expansion in velocity, and we want the zero-order term in that expansion to be close to how the function looks near $t = 0$, and another way of putting this is that we want the zero-order term to actually *be* the initial velocity when $t = 0$. At $t = 0$, terms of order greater than zero ought to vanish, so we declare that the first and second order terms be zero at $t = 0$.

We can just integrate to get the zero-order term, so $v_0 = \frac{t}{\tau} + k$, with k a constant. We want this to fit the BC above, so $k = \frac{v_i}{v_t}$. Therefore
$$\left(\frac{v}{v_t}\right)_0 = \frac{t}{\tau} + \frac{v_i}{v_t}$$

Next, we can get the first-order term by plugging in the zero-order term and integrating, so, using the lazy notation for a moment we have
$$v_1' = -\frac{1}{\tau}v_0^2 = -\frac{1}{\tau}\left(\frac{t}{\tau} + v_i\right)^2 = -\frac{1}{\tau}\left(\frac{t^2}{\tau^2} + 2\frac{v_i t}{\tau} + v_i^2\right) = -\frac{t^2}{\tau^3} - 2\frac{v_i t}{\tau^2} - \frac{v_i^2}{\tau}$$
So by integrating, we get
$$v_1 = -\frac{t^3}{3\tau^3} - \frac{v_i t^2}{\tau^2} - \frac{v_i^2 t}{\tau} + k$$
and since $v_1(0) = 0$, $k = 0$ so
$$v_1 = -\frac{t^3}{3\tau^3} - \frac{v_i t^2}{\tau^2} - \frac{v_i^2 t}{\tau}$$
Finally, we do the same kind of thing to get the second order term, so
$$v_2' = -\frac{2}{\tau}v_0 v_1 = -\frac{2}{\tau}\left(\frac{t}{\tau} + v_i\right)\left(-\frac{t^3}{3\tau^3} - \frac{v_i t^2}{\tau^2} - \frac{v_i^2 t}{\tau}\right)$$

And now after some nasty algebra,

$$v_2' = 2\left(\frac{t^4}{3\tau^5} + \frac{4v_i t^3}{3\tau^4} + \frac{2v_i^2 t^2}{\tau^3} + \frac{v_i^3 t}{\tau^2}\right)$$

and after we integrate, with the constant of integration being zero since $v_2(0) = 0$

$$v_2 = \frac{2t^5}{15\tau^5} + \frac{2v_i t^4}{3\tau^4} + \frac{4v_i^2 t^3}{3\tau^3} + \frac{v_i^3 t^2}{\tau^2}$$

Switching back to the original notation and summing up, we've found that

$$\left(\frac{v}{v_t}\right)_0 = \frac{t}{\tau} + \left(\frac{v_i}{v_t}\right)$$

$$\left(\frac{v}{v_t}\right)_1 = -\frac{t^3}{3\tau^3} - \left(\frac{t}{\tau}\right)^2\left(\frac{v_i}{v_t}\right) - \left(\frac{v_i}{v_t}\right)^2\left(\frac{t}{\tau}\right)$$

$$\left(\frac{v}{v_t}\right)_2 = \frac{2t^5}{15\tau^5} + \left(\frac{v_i}{v_t}\right)\frac{2t^4}{3\tau^4} + \left(\frac{v_i}{v_t}\right)^2\frac{4t^3}{3\tau^3} + \left(\frac{v_i}{v_t}\right)^3\frac{t^2}{\tau^2}$$