PHYS 374. Homework # 11. Young Soo Yoon.

1. a) $\rho(r,\theta) = A\cos^2\theta\, r^2 \exp\left(-\frac{r^2}{L^2}\right)$

$\rho(r,\theta) = \sum_l \rho_l(r) P_l(\cos\theta)$ where $\rho_l = \frac{2l+1}{2}\int_0^\pi P_l(\cos\theta)\rho(r,\theta)\sin\theta\, d\theta$

$P_0(\cos\theta) = 1$, $P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$

$\cos^2\theta = \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)$

$\rho(r,\theta) = A\left(\frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)\right) r^2 \exp\left(-\frac{r^2}{L^2}\right)$

$\rho_l(r) = \frac{2l+1}{2}\int_0^\pi P_l(\cos\theta)\, A\left(\frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)\right) r^2 \exp\left(-\frac{r^2}{L^2}\right)\sin\theta\, d\theta$

$= A r^2 \exp\left(-\frac{r^2}{L^2}\right)\left[\frac{2}{3}\delta_{l,2} + \frac{1}{3}\delta_{l,0}\right]$

$\therefore\ \rho_0 = \frac{1}{3}Ar^2\exp\left(-\frac{r^2}{L^2}\right)$

$\rho_2 = \frac{2}{3}Ar^2\exp\left(-\frac{r^2}{L^2}\right)$ //

b) $\Phi_l(r) = \frac{4\pi}{2l+1}\frac{1}{r^{l+1}}\int_0^r \rho_l(r')r'^{\,l+2}dr' - \frac{4\pi}{2l+1}r^l\int_r^\infty \frac{\rho_l(r')}{r'^{\,(l-1)}}dr'$

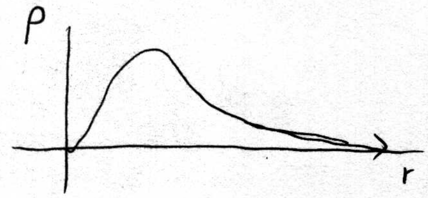
$\Phi_0(r) = \frac{4\pi}{r}\int_0^r dr' \frac{A}{3}r'^2 e^{-\frac{r'^2}{L^2}}\cdot r'^2 dr' - 4\pi\int_r^\infty dr' \frac{1}{3}Ar^2 e^{-\frac{r'^2}{L^2}}\cdot r'$

$\Phi_2(r) = \frac{4\pi}{5r^3}\int_0^r dr' \frac{2}{3}Ar'^2\exp\left(-\frac{r'^2}{L^2}\right)\cdot r'^4 - \frac{4\pi}{5}r^2\int_r^\infty \frac{\frac{2}{3}Ar'^2}{r'}\exp\left(-\frac{r'^2}{L^2}\right)dr'$

$\Phi(r,\theta) = \Phi_0 P_0(\cos\theta) + \Phi_2(r)P_2(\cos\theta)$

* Another possible answer,

Assume $r \gg L$,



for $r \gg L$, we can use the solution of Laplace Equation

because $e^{-\frac{r^2}{L^2}}$ is extremely small

$$\Phi(r,\theta) = \sum_l \frac{a_l}{r^{l+1}} P_l(\cos\theta)$$

$$a_l = \frac{4\pi}{2l+1} \int_0^\infty dr' \rho_l(r') \cdot r'^{(l+2)}$$

$$a_0 = \frac{4\pi}{1} \int_0^\infty \rho_0(r') \cdot r'^2 \cdot dr' = 4\pi \int_0^\infty dr' \frac{1}{3} A r'^2 \cdot \exp\left(-\frac{r'^2}{L^2}\right) \cdot r'^2$$

$$= 4\pi \cdot \frac{A}{\cancel{3}} \cdot \frac{\cancel{3}}{8} L^5 \sqrt{\pi} \qquad \text{(from mathmatica)}$$

$$= \frac{1}{2} A\pi L^5 \sqrt{\pi}$$

$$a_2 = \frac{4\pi}{5} \int_0^\infty dr' \rho_2(r') \cdot r'^4$$

$$= \frac{4\pi}{5} \int_0^\infty dr' \frac{2}{3} A r'^2 \exp\left(-\frac{r^2}{L^2}\right) r'^4$$

$$= \frac{4\pi}{\cancel{5}} \cdot \frac{2}{\cancel{3}} A \cdot \frac{\cancel{15}}{16} L^7 \sqrt{\pi} = \frac{\pi}{2} A L^7 \sqrt{\pi}$$

$$\Phi(r,\theta) = \frac{\pi}{2} A L^5 \frac{\sqrt{\pi}}{r} P_0(\cos\theta) + \frac{\pi}{2} A L^7 \frac{\sqrt{\pi}}{r^3} P_2(\cos\theta)$$

2. $f(t) = A \exp\left(-\frac{t^2}{\tau^2}\right)$.

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt\ f(t)\, e^{i\omega t} = \int_{-\infty}^{\infty} dt\ A \exp\left[-\frac{t^2}{\tau^2} + i\omega t\right]$$

$$= \int_{-\infty}^{\infty} dt\ A \exp\left[-\frac{1}{\tau^2}\left(t - \frac{i\omega\tau^2}{2}\right)^2 - \frac{\omega^2\tau^2}{4}\right]$$

$$t' = t - \frac{i\omega\tau}{2} \qquad dt' = dt.$$

$$\tilde{f}(\omega) = A \exp\left(-\frac{\omega^2\tau^2}{4}\right) \underbrace{\int_{-\infty}^{\infty} dt' \exp\left[-\frac{1}{\tau^2} t'^2\right]}$$

from the hint in the problem.

$$= A \exp\left(-\frac{\omega^2\tau^2}{4}\right) \cdot \tau\sqrt{\pi}$$

$$\therefore\ \tilde{f}(\omega) = \tau\sqrt{\pi}\, A \exp\left(-\frac{\omega^2\tau^2}{4}\right)$$

3. $$\hat{A} \equiv \frac{1}{1 - a\frac{d}{dt}} = 1 + a\frac{d}{dt} + \left(a\frac{d}{dt}\right)^2 + \left(a\frac{d}{dt}\right)^3 + \cdots$$

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega),$$

$$\hat{A} f(t) = f(t) + a\frac{d}{dt} f(t) + \left(a\frac{d}{dt}\right)^2 f(t) + \left(a\frac{d}{dt}\right)^3 f(t) + \cdots$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega) + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega)(-i\omega\cdot a)$$

$$+ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega)(-i\omega a)^2 + \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega)(-i\omega a)^3 + \cdots$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega) \cdot \left(1 + (-i\omega a) + (-i\omega a)^2 + (-i\omega a)^3 + \cdots\right)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{f}(\omega) \cdot \frac{1}{1 + ia\omega}$$