

1. a) an approximately elliptical conductor. $R_1(\phi) = \sqrt{\alpha^2 \cos^2 \phi + \beta^2 \sin^2 \phi}$

The general expression of the potential,

$$\Phi(r, \phi) = a_0 + b_0 \log\left(\frac{r}{r_0}\right) + \sum_m \left(\frac{a_m}{r^m} + b_m r^m \right) e^{im\phi} + C.C.$$

The surface of the conductor is held at potential V,

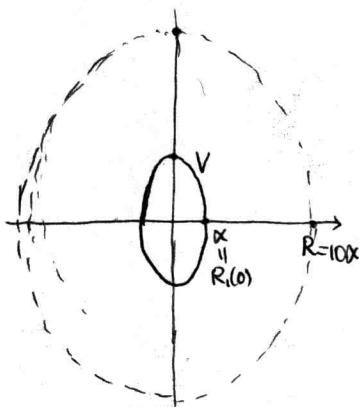
$$R_{1s}^2(\phi_s) = \alpha^2 \cos^2 \phi_s + \beta^2 \sin^2 \phi_s = r_s^2$$

$$\begin{aligned} \Phi(r_s, \phi_s) &= a_0 + b_0 \log\left(\frac{r_s}{r_0}\right) + \sum_m \left(\frac{a_m}{r_s^m} + b_m r_s^m \right) e^{im\phi_s} + C.C. \\ &= V = V \cdot \frac{\alpha^2 \cos^2 \phi_s + \beta^2 \sin^2 \phi_s}{r_s^2} \\ &= \frac{V}{r_s^2} \left(\frac{1}{2}(\alpha^2 + \beta^2) + \frac{1}{2} \cos 2\phi_s - \frac{1}{2} \cos 2\phi_s \right) \end{aligned}$$

\Rightarrow only $m=0$ and $m=2$ terms exist.

So we can say,

$$\Phi(r, \phi) = a_0 + b_0 \log\left(\frac{r}{r_0}\right) + \frac{a_2}{r^2} \cos 2\phi + b_2 r^2 \cos 2\phi$$



$$\beta = 1.2 \alpha \text{ and } R = 10 \alpha$$

$$\text{at } r = \alpha \text{ and } \phi = 0$$

$$V = a_0 + b_0 \log\left(\frac{\alpha}{r_0}\right) + \frac{a_2}{\alpha^2} + b_2 \cdot \alpha^2$$

$$\text{at } r = \beta \text{ and } \phi = \frac{\pi}{2}$$

$$V = a_0 + b_0 \log\left(\frac{\beta}{r_0}\right) + \frac{a_2}{\beta^2} \cdot (-1) + b_2 \cdot \beta^2 \cdot (-1)$$

$$\text{at } r = R, \phi = 0, \quad 0 = a_0 + b_0 \log\left(\frac{R}{r_0}\right) + \frac{a_2}{R^2} + b_2 R^2$$

$$\text{at } r = R, \phi = \frac{\pi}{2}, \quad 0 = a_0 + b_0 \log\left(\frac{R}{r_0}\right) - \frac{a_2}{R^2} - b_2 R^2$$

$$\Rightarrow V = a_0 + b_0 \log \frac{\alpha}{r_0} + \frac{a_2}{\alpha^2} + b_2 \alpha^2 \quad \text{--- ①}$$

$$V = a_0 + b_0 \log \frac{1.2\alpha}{r_0} - \frac{a_2}{(1.2\alpha)^2} - b_2 (1.2\alpha)^2 \quad \text{--- ②}$$

$$0 = a_0 + b_0 \log \frac{10\alpha}{r_0} + \frac{a_2}{100\alpha^2} + 100b_2 \alpha^2 \quad \text{--- ③}$$

$$0 = a_0 + b_0 \log \frac{10\alpha}{r_0} - \frac{a_2}{100\alpha^2} - 100b_2 \alpha^2 \quad \text{--- ④}$$

$$\textcircled{3} + \textcircled{4} \quad 2(a_0 + b_0 \log \frac{10\alpha}{r_0}) = 0$$

$$\textcircled{3} - \textcircled{4} \quad 2 \left(\frac{a_2}{100\alpha^2} + 100b_2 \alpha^2 \right) = 0$$

① + ②

$$\begin{aligned} 2V &= \underbrace{2a_0 + 2b_0 \log \frac{\alpha}{r_0}}_{-2b_0 \log 10} + b_0 \log 1.2 + \frac{a_2}{\alpha^2} \left(1 - \frac{1}{1.44} \right) + b_2 \alpha^2 \left(1 - 1.44 \right) \\ &= -2b_0 \log 10 + b_0 \log 1.2 + \frac{a_2}{\alpha^2} \frac{0.44}{1.44} + b_2 \alpha^2 (-0.44) \\ &= b_0 \log \frac{1.2}{100} + \frac{a_2}{\alpha^2} \frac{0.44}{1.44} + b_2 \alpha^2 (-0.44) \quad \text{--- ⑤} \end{aligned}$$

① - ②

$$0 = -b_0 \log 1.2 + \frac{a_2}{\alpha^2} \left(1 + \frac{1}{1.44} \right) + b_2 \alpha^2 (1 + 1.44) \quad \text{--- ⑥}$$

From ⑤

$$\begin{aligned} 2V &= b_0 \log \frac{1.2}{100} + \frac{a_2}{\alpha^2} \frac{0.44}{1.44} + (-0.44) \left(-\frac{a_2}{100^2 \alpha^2} \right) \\ &= b_0 \log \frac{1.2}{100} + \frac{a_2}{\alpha^2} \left(\frac{0.44}{1.44} + \frac{0.44}{100^2} \right) \approx -4.42 b_0 + \frac{a_2}{\alpha^2} 0.3056 \end{aligned}$$

From ⑥

$$\begin{aligned} b_0 \log 1.2 &= \frac{a_2}{\alpha^2} \frac{2.44}{1.44} + 2.44 \times \left(-\frac{a_2}{100^2 \alpha^2} \right) \\ &= \frac{a_2}{\alpha^2} \left(\frac{2.44}{1.44} - \frac{2.44}{100^2} \right) \end{aligned}$$

$$0.182 b_0 \approx \boxed{+1.6942} \frac{a_2}{\alpha^2}$$

$$b_0 = -0.4558 V$$

$$a_2 = -0.04897 V \alpha^2$$

$$a_0 = -b_0 \log \frac{10X}{r_0} = 0.4558 V \log \frac{10X}{r_0}$$

$$b_2 = -\frac{a_2}{100^2 \alpha^4} = +0.04897 \times 10^{-4} V \frac{1}{\alpha^2}$$

* If you use, $\Phi = a'_0 + b'_0 \log(\frac{r}{r_0}) + \frac{2a_2}{r^2} \cos 2\phi + 2b_2 \cos 2\phi$,

the final potential will be same. ($\Phi(r, \phi)$) referring the result of the mathematica.

$$\Rightarrow \Phi(r, \phi) = +0.4558 V \log \frac{10X}{r_0} - 0.4558 V \log \left(\frac{r}{r_0} \right)$$

$$+ 0.04897 V \alpha^2 \frac{1}{r^2} \cos 2\phi - 0.04897 \times 10^{-4} \frac{V}{\alpha^2} r^2 \cos 2\phi$$

$$= 0.4558 V \log \left(\frac{10X}{r} \right) + 0.04897 V \alpha^2 \frac{\cos 2\phi}{r^2}$$

$$- 0.04897 \times 10^{-4} \frac{V}{\alpha^2} r^2 \cos^2 2\phi$$

$$(2) r^2 = x^2 + y^2, \quad \tan \phi = \frac{y}{x}$$

$$\cos 2\phi = \cos^2 \phi - \sin^2 \phi = \cos^2 \phi (1 - \tan^2 \phi)$$

$$= \frac{\cos^2 \phi}{\sin^2 \phi + \cos^2 \phi} (1 - \tan^2 \phi) = \frac{1 - \tan^2 \phi}{\tan^2 \phi + 1} = \frac{1 - \left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)^2} = \frac{x^2 - y^2}{x^2 + y^2}$$

$$r^2 \cos 2\phi = x^2 - y^2, \quad \frac{\cos 2\phi}{r^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\Phi(x, y) = 0.4558 V \log(10X) - \frac{0.4558}{2} V \log(x^2 + y^2)$$

$$+ 0.04897 V \alpha^2 \frac{x^2 - y^2}{(x^2 + y^2)^2} - 0.04897 \times 10^{-4} \frac{V}{\alpha^2} (x^2 - y^2)$$

using Mathematica

$$a_0 = 0.4558 V \log \frac{10X}{r_0}$$

$$b_0 = -0.4558 V$$

$$a_2 = -0.04897 V \alpha^2$$

$$b_2 = 4.90278 \times 10^{-6} V \frac{1}{\alpha^2}$$

Prob 1. (b) set $r_0 = \alpha$,

In[1]:= { a_0, b_0, a_2, b_2 } =

$$\text{Inverse} \left[\begin{pmatrix} 1 & \text{Log} \left[\frac{\alpha}{\alpha} \right] & \frac{2}{\alpha^2} & 2 \alpha^2 \\ 1 & \text{Log} \left[\frac{1.2 \alpha}{\alpha} \right] & \frac{-2}{(1.2 \alpha)^2} & -2 (1.2 \alpha)^2 \\ 1 & \text{Log} \left[\frac{10 \alpha}{\alpha} \right] & \frac{2}{100 \alpha^2} & 200 \alpha^2 \\ 1 & \text{Log} \left[\frac{10 \alpha}{\alpha} \right] & \frac{-2}{100 \alpha^2} & -200 \alpha^2 \end{pmatrix} \right] \cdot \begin{pmatrix} V \\ V \\ 0 \\ 0 \end{pmatrix} // \text{FullSimplify}$$

$$\text{Out}[1] = \left\{ \{1.04902 \text{ V}\}, \{-0.455585 \text{ V}\}, \{-0.0245139 \text{ V} \alpha^2\}, \left\{ \frac{2.45139 \times 10^{-6} \text{ V}}{\alpha^2} \right\} \right\}$$

$$\text{In}[2]:= \Phi[r, \phi] = a_0 + b_0 \text{Log} \left[\frac{r}{\alpha} \right] + 2 \frac{a_2}{r^2} \cos[2\phi] + 2 b_2 r^2 \cos[2\phi]$$

$$\text{Out}[2] = \left\{ 1.04902 \text{ V} + \frac{4.90278 \times 10^{-6} r^2 \text{ V} \cos[2\phi]}{\alpha^2} - \frac{0.0490278 \text{ V} \alpha^2 \cos[2\phi]}{r^2} - 0.455585 \text{ V} \text{Log} \left[\frac{r}{\alpha} \right] \right\}$$

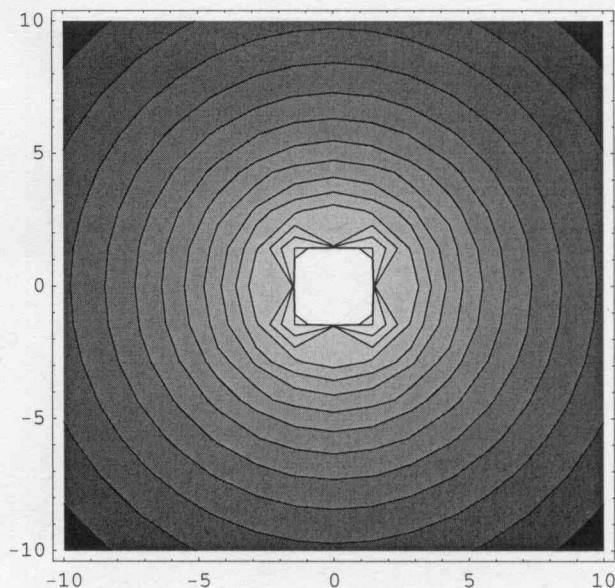
Prob 1. (c)

In[1]:= V = 10; $\alpha = 1$;

$$\begin{aligned} \Phi[x, y] = & 0.449 \text{ V} * \text{Log}[10 \alpha] - 0.449 \text{ V} * \text{Log}[\sqrt{x^2 + y^2}] + \\ & 0.04825 \text{ V} \alpha^2 \frac{x^2 - y^2}{(x^2 + y^2)^2} - 0.04825 * 10^{-4} * (x^2 - y^2) \end{aligned}$$

$$\text{Out}[1] = 10.3386 - 4.825 \times 10^{-6} (x^2 - y^2) + \frac{0.4825 (x^2 - y^2)}{(x^2 + y^2)^2} - 4.49 \text{ Log}[\sqrt{x^2 + y^2}]$$

In[2]:= ContourPlot[\Phi[x, y], {x, -10, 10}, {y, -10, 10}, Contours -> 15];



Prob 2. (a)

In[1]:= {aa₀, bb₀, aa₂, bb₂} =

$$\text{Inverse} \left[\begin{pmatrix} 1 & \text{Log} \left[\frac{\sqrt{1.11} \alpha}{\alpha} \right] & \frac{2}{2*1.11 \alpha^2} & 2 * \frac{1.11}{2} \alpha^2 \\ 1 & \text{Log} \left[\frac{\sqrt{1.33} \alpha}{\alpha} \right] & \frac{-1}{1.33 \alpha^2} & -2 * \frac{1.33}{2} \alpha^2 \\ 1 & \text{Log} \left[\frac{10 \alpha}{\alpha} \right] & \frac{2}{200 \alpha^2} & \frac{200}{2} \alpha^2 \\ 1 & \text{Log} \left[\frac{10 \alpha}{\alpha} \right] & \frac{-2}{200 \alpha^2} & \frac{-200}{2} \alpha^2 \end{pmatrix} \right] . \begin{pmatrix} V \\ V \\ 0 \\ 0 \end{pmatrix} // \text{FullSimplify}$$

Out[1]= $\left\{ \{1.0461 \text{V}\}, \{-0.454313 \text{V}\}, \{-0.0248553 \text{V} \alpha^2\}, \left\{ \frac{2.48553 \times 10^{-6} \text{V}}{\alpha^2} \right\} \right\}$

Prob 2. (b)

In[2]:= {aaa₀, bbb₀, aaa₂, bbb₂, aaa₄, bbb₄} =

$$\text{Inverse} \left[\begin{pmatrix} 1 & \text{Log} \left[\frac{\alpha}{\alpha} \right] & \frac{2}{\alpha^2} & 2 \alpha^2 & \frac{2}{\alpha^4} & 2 \alpha^4 \\ 1 & \text{Log} \left[\frac{\sqrt{1.22} \alpha}{\alpha} \right] & 0 & 0 & \frac{-2}{(1.22)^2 \alpha^4} & -2 (1.22)^2 \alpha^4 \\ 1 & \text{Log} \left[\frac{1.2 \alpha}{\alpha} \right] & \frac{-2}{(1.2 \alpha)^2} & -2 (1.2 \alpha)^2 & \frac{2}{(1.2 \alpha)^4} & 2 (1.2 \alpha)^4 \\ 1 & \text{Log} \left[\frac{10 \alpha}{\alpha} \right] & \frac{2}{(10 \alpha)^2} & 2 (10 \alpha)^2 & \frac{2}{(10 \alpha)^4} & 2 (10 \alpha)^4 \\ 1 & \text{Log} \left[\frac{10 \alpha}{\alpha} \right] & 0 & 0 & \frac{-2}{(10 \alpha)^4} & -2 (10 \alpha)^4 \\ 1 & \text{Log} \left[\frac{10 \alpha}{\alpha} \right] & \frac{-2}{(10 \alpha)^2} & -2 (10 \alpha)^2 & \frac{2}{(10 \alpha)^4} & 2 (10 \alpha)^4 \end{pmatrix} \right] . \begin{pmatrix} V \\ V \\ V \\ 0 \\ 0 \\ 0 \end{pmatrix} //$$

FullSimplify

Out[2]= $\left\{ \{1.04704 \text{V}\}, \{-0.454724 \text{V}\}, \{-0.0248835 \text{V} \alpha^2\}, \left\{ \frac{2.48835 \times 10^{-6} \text{V}}{\alpha^2} \right\}, \{0.00136103 \text{V} \alpha^4\}, \left\{ -\frac{1.36103 \times 10^{-11} \text{V}}{\alpha^4} \right\} \right\}$

c) $\vec{E} = -\nabla \Phi(x, y) = -(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}) \Phi(x, y)$

$$= - \left[-\frac{1}{2} \times 0.4558 \frac{\frac{\partial x}{x^2+y^2}}{\alpha^2} V + 0.4897 V \alpha^2 \left(\frac{2x}{(x^2+y^2)^2} + (x^2-y^2) \frac{-2 \cdot 2x}{(x^2+y^2)^3} \right) \right. \\ \left. - 0.04897 \times 10^{-4} \frac{V}{\alpha^2} \cdot 2x \right] \hat{x}$$

$$- \left[-\frac{1}{2} \times 0.4558 \frac{\frac{\partial x}{x^2+y^2}}{\alpha^2} \cdot V + 0.4897 V \alpha^2 \left(\frac{-2y}{(x^2+y^2)^2} + (x^2-y^2) \frac{-2 \cdot 2y}{(x^2+y^2)^3} \right) \right. \\ \left. + 0.04897 \times 10^{-4} \frac{V}{\alpha^2} \cdot (-2y) \right] \hat{y}$$

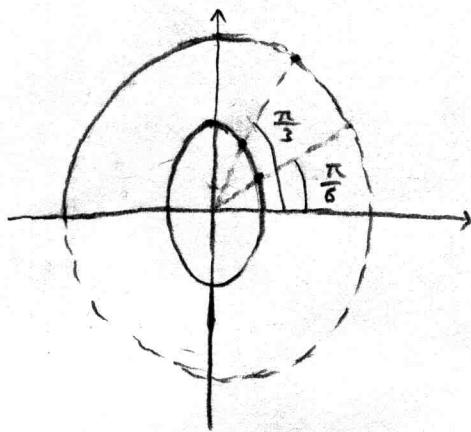
$$= \left[0.4558 \frac{\frac{\partial}{x^2+y^2}}{\alpha^2} - \frac{0.4897}{(x^2+y^2)^3} \alpha^2 (2x^3+2xy^2 - 4x^3+4xy^2) \right. \\ \left. + 2 \times 0.04897 \times 10^{-4} \frac{\frac{\partial}{\alpha^2}}{\alpha^2} \right] V \hat{x}$$

$$+ \left[0.4558 \frac{\frac{\partial}{x^2+y^2}}{\alpha^2} + \frac{0.4897}{(x^2+y^2)^3} \alpha^2 (2yx^2+2y^3 + 4yx^2 - 4y^3) \right. \\ \left. - 2 \times 0.04897 \times 10^{-4} \frac{\frac{\partial}{\alpha^2}}{\alpha^2} \right] V \hat{y}$$

$$= \left[0.4558 \frac{\frac{\partial}{x^2+y^2}}{\alpha^2} + 0.4897 \alpha^2 \frac{2x^3-6xy^2}{(x^2+y^2)^3} + 9.794 \times 10^{-6} \frac{\frac{\partial}{\alpha^2}}{\alpha^2} \right] V \hat{x}$$

$$+ \left[0.4558 \frac{\frac{\partial}{x^2+y^2}}{\alpha^2} + 0.4897 \alpha^2 \frac{6x^2y-2y^3}{(x^2+y^2)^3} - 9.794 \times 10^{-6} \frac{\frac{\partial}{\alpha^2}}{\alpha^2} \right] V \hat{y}$$

2. a) Choose the points at $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$.



$$\text{at } \frac{\pi}{6} = \phi_1', \quad r_{s_1}^2 = \alpha^2 \cos^2 \frac{\pi}{6} + \beta^2 \sin^2 \frac{\pi}{6} \\ = \frac{3}{4} \alpha^2 + \frac{1}{4} \beta^2 = \frac{3 + (1.2)^2}{4} \alpha^2 = \alpha_1'^2$$

$$\text{at } \frac{\pi}{3} = \phi_2', \quad r_{s_2}^2 = \alpha^2 \cos^2 \frac{\pi}{3} + \beta^2 \sin^2 \frac{\pi}{3} = \frac{1}{4} \alpha^2 + \frac{3}{4} \beta^2 \\ = \alpha_2'^2$$

$$\Phi(r, \phi) = a_0 + b_0 \log\left(\frac{r}{r_0}\right) + \frac{a_2}{r^2} \cos 2\phi + b_2 r^2 \cos 2\phi$$

At $r = \alpha_1'$ and $\phi = \frac{\pi}{6}$,

$$\Phi(\alpha_1', \frac{\pi}{6}) = V = a_0 + b_0 \log \frac{\alpha_1'}{r_0} + \frac{a_2}{\alpha_1'^2} \left(+ \frac{1}{2} \right) + b_2 \alpha_1'^2 \left(+ \frac{1}{2} \right)$$

At $r = \alpha_2'$, $\phi = \frac{\pi}{3}$,

$$\Phi(\alpha_2', \frac{\pi}{3}) = V = a_0 + b_0 \log \frac{\alpha_2'}{r_0} + \frac{a_2}{\alpha_2'^2} \left(- \frac{1}{2} \right) + b_2 \alpha_2'^2 \left(- \frac{1}{2} \right)$$

At $r = 10\alpha$, $\phi = \frac{\pi}{3}$,

$$\Phi(10\alpha, \frac{\pi}{3}) = 0 = a_0 + b_0 \log \frac{10\alpha}{r_0} + \frac{1}{2} \frac{a_2}{(10\alpha)^2} + b_2 (10\alpha)^2 \frac{1}{2}$$

At $r = 10\alpha$, $\phi = \frac{\pi}{6}$

$$\Phi(10\alpha, \frac{\pi}{6}) = 0 = a_0 + b_0 \log \frac{10\alpha}{r_0} + \frac{1}{2} \cdot \frac{a_2}{(10\alpha)^2} + b_2 (10\alpha)^2 \left(- \frac{1}{2} \right)$$

$$\Rightarrow \begin{pmatrix} V \\ V \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & \log \frac{\alpha_1'}{r_0} & + \frac{1}{2\alpha_1'^2} & \frac{\alpha_1'^2}{2} \\ 1 & \log \frac{\alpha_2'}{r_0} & - \frac{1}{2\alpha_2'^2} & - \frac{\alpha_2'^2}{2} \\ 1 & \log \frac{10\alpha}{r_0} & \frac{1}{200\alpha^2} & 50\alpha^2 \\ 1 & \log \frac{10\alpha}{r_0} & - \frac{1}{200\alpha^2} & - 50\alpha^2 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ a_2 \\ b_2 \end{pmatrix}$$

$$\left(\text{where } \alpha_1' = \sqrt{\frac{4.44}{4}} \alpha = \sqrt{1.11} \alpha, \quad \alpha_2' = \sqrt{\frac{5.32}{4}} \alpha = \sqrt{1.33} \alpha \right)$$

Using Mathematica,

$$\Rightarrow \begin{cases} a_0 = 0.4534 \sqrt{\log\left(\frac{10x}{r_0}\right)} \\ b_0 = -0.4534 \sqrt{\alpha^2} \\ a_2 = -0.04551 \sqrt{\alpha^2} \\ b_2 = 4.551 \times 10^{-6} \frac{\sqrt{\alpha^2}}{\alpha^2} \end{cases}$$

These are almost same to the results of prob 1.

b)

$$\bar{E}(r, \phi) = a_0 + b_0 \log \frac{r}{r_0} + \frac{a_2}{r^2} \cos 2\phi + b_2 r^2 \cos 2\phi + \frac{a_4}{r^4} \cos 4\phi + b_4 r^4 \cos 4\phi$$

at $r = \alpha, \phi = 0$,

$$V = a_0 + b_0 \log \frac{\alpha}{r_0} + \frac{a_2}{\alpha^2} + b_2 \alpha^2 + \frac{a_4}{\alpha^4} + b_4 \alpha^4$$

$$\text{at } r = \alpha_1, \phi = \frac{\pi}{4} \rightarrow r_s^2 = \alpha^2 \left(\frac{1}{\sqrt{2}}\right)^2 + \beta^2 \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}(1 + (1.2)^2) \alpha^2 = 1.22 \alpha^2 = \alpha_1^2$$

$$V = a_0 + b_0 \log \frac{\sqrt{1.22}}{r_0} \alpha + \frac{a_2}{1.22 \alpha^2} \times 0 + b_2 \times 1.22 \alpha^2 \times 0 + \frac{a_4}{(1.22)^2 \alpha^4} (-1) + b_4 (1.22 \alpha^2)^4$$

$$\text{at } r = \beta, \phi = \frac{\pi}{2} \rightarrow r_s = \beta = 1.2 \alpha$$

$$V = a_0 + b_0 \log \frac{1.2 \alpha}{r_0} + \frac{(-1)a_2}{(1.2 \alpha)^2} + b_2 (1.2 \alpha)^2 \times (-1) + \frac{a_4}{(1.2 \alpha)^4} + b_4 (1.2 \alpha)^4$$

$$\text{at } r = 10 \alpha, \phi = 0.$$

$$O = a_0 + b_0 \log \frac{10\alpha}{r_0} + \frac{a_2}{(10\alpha)^2} + b_2 (10\alpha)^2 + \frac{a_4}{(10\alpha)^4} + b_4 (10\alpha)^4$$

$$\text{at } r = 10 \alpha, \phi = \frac{\pi}{4}$$

$$O = a_0 + b_0 \log \frac{10\alpha}{r_0} + O + O + \frac{a_4}{(10\alpha)^4} \times (-1) + b_4 (10\alpha)^4 \cdot (-1)$$

$$\text{at } r = 10 \alpha, \phi = \frac{\pi}{2}$$

$$O = a_0 + b_0 \log \frac{10\alpha}{r} + \frac{a_2}{(10\alpha)^2} \times (-1) + b_2 (10\alpha)^2 \times (-1) + \frac{a_4}{(10\alpha)^4} + b_4 (10\alpha)^4$$

$$\Rightarrow \begin{pmatrix} V \\ V \\ V \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & \log \frac{\alpha}{r_0} & \frac{1}{\alpha^2} & \alpha^2 & \frac{1}{\alpha^4} & \alpha^4 \\ 1 & \log \frac{1.2\alpha}{r_0} & 0 & 0 & \frac{-1}{(1.2\alpha)^2 \alpha^4} & -(1.2\alpha)^2 \alpha^4 \\ 1 & \log \frac{1.2\alpha}{r_0} & \frac{-1}{(1.2\alpha)^2} & -(\frac{1}{2}\alpha)^2 & \frac{1}{(1.2\alpha)^4} & (1.2\alpha)^4 \\ 1 & \log \frac{10\alpha}{r_0} & \frac{1}{(10\alpha)^2} & (10\alpha)^2 & \frac{1}{(10\alpha)^4} & (10\alpha)^4 \\ 1 & \log \frac{10\alpha}{r_0} & 0 & 0 & -\frac{1}{(10\alpha)^4} & -(10\alpha)^4 \\ 1 & \log \frac{10\alpha}{r_0} & \frac{-1}{(10\alpha)^2} & -(\frac{1}{10}\alpha)^2 & \frac{1}{(10\alpha)^4} & (10\alpha)^4 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ a_2 \\ b_2 \\ a_4 \\ b_4 \end{pmatrix}$$

Using Mathematica,

~~$$a_0 = (0.1403 + 0.05642 \log \frac{\alpha}{r_0}) V$$~~

~~$$b_0 = -0.03642 V$$~~

~~$$b_2 = \frac{V}{\alpha^2} \left(-0.4234 \times 10^{-3} - 3.0882 \times 10^{-8} \log \left(\frac{\alpha}{r_0} \right) \right)$$~~

~~$$a_2 = 2.1479 V \alpha^2$$~~

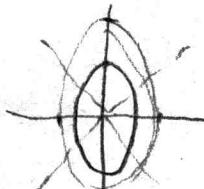
~~$$a_4 = V \alpha^4 \left(-1.2878 + 1.9460 \times 10^{-15} \log \left(\frac{\alpha}{r_0} \right) \right)$$~~

~~$$b_4 = \frac{V}{\alpha^4} \left(1.0564 \times 10^{-6} + 6.1244 \times 10^{-20} \log \left(\frac{\alpha}{r_0} \right) \right)$$~~

c) When $\beta = 1.4\alpha$, the effect of the coefficient with $m=4$, is bigger than that of $\beta = 1.2\alpha$.

If $\alpha = \beta$, it's symmetric, so it only depends on $m=2$. ($\alpha \sim \frac{1}{r^2}$)

If β is getting bigger than α , the effect of r^4 or r^{-4} terms
• is getting bigger.



3. On the surface of a sphere of radius R , $\Phi(R, \theta) = V_0 \cos^2 \theta$

a) Using the known Legendre polynomials,

$$P_0(x) = 1 \rightarrow P_0(\cos \theta) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \rightarrow P_2(\cos \theta) = \frac{1}{2}(3\cos^2 \theta - 1)$$

$$\rightarrow \cos^2 \theta = \frac{1}{3}(2P_2(\cos \theta) + 1) = \frac{1}{3}(2P_2(\cos \theta) + P_0(\cos \theta))$$

$$\therefore \Phi(R, \theta) = \frac{2}{3}V_0 P_2(\cos \theta) + \frac{1}{3}V_0 P_0(\cos \theta)$$

b) for $r > R$,

The general solution of $\Phi(r, \theta)$ using Legendre polynomials,

$$\Phi(r, \theta) = \sum_l \left(\frac{a_l}{r^{l+1}} + b_l r^l \right) P_l(\cos \theta)$$

when $r \rightarrow \infty$, r^l will diverge. so, b_l should be 0.

On the boundary, $r = R$.

$$\Phi(R, \theta) = \frac{2}{3}V_0 P_2(\cos \theta) + \frac{1}{3}V_0 P_0(\cos \theta) = \frac{a_2}{R^3} P_2(\cos \theta) + \frac{a_0}{R} P_0(\cos \theta)$$

$$a_2 = \frac{2}{3}V_0 R^3, \quad a_0 = \frac{1}{3}RV_0$$

$$\therefore \Phi(r, \theta) = \frac{2}{3}V_0 \frac{R^3}{r^3} P_2(\cos \theta) + \frac{V_0}{3} \frac{R}{r} P_0(\cos \theta)$$

$$\vec{E} = -\nabla \Phi(r, \theta) = -\left(\frac{\partial}{\partial r} \hat{r} + \frac{i}{r \sin \theta} \frac{\partial}{\partial \theta} \hat{\theta}\right) \Phi(r, \theta) \quad (\text{in spherical coordinate})$$

$$= -\left(\frac{\partial}{\partial r} \hat{r} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \hat{\theta}\right) \left(\frac{2V_0 R^3}{3r^3} \frac{1}{r^3} (3\cos^2 \theta - 1) + \frac{V_0 R}{3r} \right)$$

$$= \left(V_0 R^3 \frac{1}{r^4} (3\cos^2 \theta - 1) + \frac{V_0 R}{3r^2}\right) \hat{r} + \frac{1}{r \sin \theta} V_0 R^3 \frac{1}{r^3} 2\cos \theta \sin \theta \hat{\theta}$$

$$\vec{E} = \frac{V_0 R^3}{r^2} \left(\frac{3\cos^2\theta - 1}{r^2} + \frac{1}{3R^2} \right) \hat{r} + \frac{2V_0 R^3}{r^4} \cos\theta \hat{\theta}$$

$$= \left(\frac{2V_0 R^3}{r^4} P_2(\cos\theta) + \frac{V_0 R}{3r^2} P_0(\cos\theta) \right) \hat{r} + \frac{2V_0 R^3}{r^4} P_1(\cos\theta) \hat{\theta}$$

c) for $r < R$, the general solution of $\Phi(r, \theta)$

$$\Phi(r, \theta) = \sum_{\ell} \left(\frac{a_{\ell}}{r^{\ell+1}} + b_{\ell} r^{\ell} \right) P_{\ell}(\cos\theta)$$

when $r \rightarrow 0$, $\frac{1}{r^{\ell+1}}$ will blow up. a_{ℓ} should be zero.

On the boundary, $\Phi(R, \theta) =$

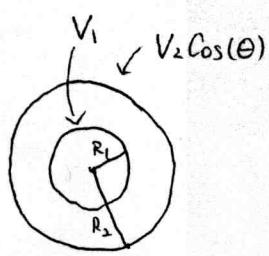
$$\Phi(R, \theta) = \frac{2}{3} V_0 P_2(\cos\theta) + \frac{1}{3} V_0 P_0(\cos\theta) = b_0 P_0(\cos\theta) + b_2 R^2 P_2(\cos\theta)$$

$$\rightarrow b_0 = \frac{1}{3} V_0 \quad \text{and} \quad b_2 = \frac{2V_0}{3R^2}$$

$$\Phi(r, \theta) = \frac{1}{3} V_0 P_0(\cos\theta) + \frac{2V_0}{3R^2} r^2 P_2(\cos\theta)$$

$$= \frac{1}{3} V_0 + \frac{2V_0}{3R^2} r^2 \frac{1}{2}(3\cos^2\theta - 1)$$

4.



$$\Phi(R_1, \theta) = V_1 \quad \text{and} \quad \Phi(R_2, \theta) = V_2 \cos \theta$$

In the region $R_1 < r < R_2$,

$$\text{In general, } \Phi(r, \theta) = \sum_l \left(\frac{a_l}{r^{l+1}} + b_l r^l \right) P_l(\cos \theta)$$

$$\text{At the boundary } \Phi(R_1, \theta) = V_1 = V_1 P_0(\cos \theta)$$

$$\Phi(R_2, \theta) = V_2 \cos \theta = V_2 P_1(\cos \theta)$$

$\Rightarrow l=0$ and $l=1$ contribute:-

$$\Phi(r, \theta) = \left(\frac{a_0}{r} + b_0 \right) P_0(\cos \theta) + \left(\frac{a_1}{r^2} + b_1 r \right) P_1(\cos \theta)$$

$$\text{At } r=R_1, \quad \Phi(R_1, \theta) = V_1$$

$$\rightarrow \underbrace{\frac{a_0}{R_1} + b_0}_{= V_1} = V_1 \quad \underbrace{\frac{a_1}{R_1^2} + b_1 R_1}_{= 0} = 0$$

$$\text{At } r=R_2, \quad \Phi(R_2, \theta) = V_2 \cos \theta$$

$$\rightarrow \underbrace{\frac{a_0}{R_2} + b_0}_{= 0} = 0 \quad \underbrace{\frac{a_1}{R_2^2} + b_1 R_2}_{= V_2} = V_2$$

$$a_0 \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = V_1 \quad a_0 = \frac{R_1 R_2 V_1}{R_2 - R_1}, \quad b_0 = -\frac{R_1 V_1}{R_2 - R_1}$$

$$a_1 \left(\frac{R_1}{R_2^2} - \frac{R_2}{R_1^2} \right) = V_2 R_1 \quad a_1 = \frac{R_1^3 R_2^2}{R_1^3 - R_2^3} V_2, \quad b_1 = \frac{R_2^2}{(R_2^3 - R_1^3)} V_2$$

$$\Rightarrow \Phi(r, \theta) = \left(\frac{R_1 R_2}{R_2 - R_1} \frac{V_1}{r} + \frac{-R_1}{R_2 - R_1} V_1 \right) + \left(\frac{-R_1^3 R_2^2}{R_2^3 - R_1^3} \frac{V_2}{r^2} + \frac{R_2^2 V_2}{R_2^3 - R_1^3} \cdot r \right) \cos \theta$$

$$= \frac{R_1 V_1}{R_2 - R_1} \left(\frac{R_2}{r} - 1 \right) P_0(\cos \theta) + \frac{R_2^2 V_2}{R_2^3 - R_1^3} \left(-\frac{R_1^3}{r^2} + r \right) P_1(\cos \theta)$$