Physics 374 ---Homework 1

In class we studied the problem of a falling body under the influence of gravity and air resistance (treated linearly in velocity). The differential equation describing this system is: $\frac{dv}{dt} = \frac{1}{\tau}\left(v_T - v\right)$ where $\tau = \frac{m}{\alpha}$ and $v_T = g\tau = \frac{mg}{\alpha}$. In class, we developed an expansion valid for small *v* (*i.e.* for $v << v_T$) which corresponds to the gravitational force dominating over air resistance.

1) Here I wish you to develop an expansion valid for small *times* (regardless of the value of *v* which implies that air resistance dominates over gravity. Again you can use λ as a counting parameter. At small times, the net effect of the force on changing the velocity is small. Thus, you can treat the entire force as being of order λ.. The differential equation can thus be represented as $\frac{dv}{dt} = \frac{\lambda}{\tau}\left(v_T - v\right)$ and the solution can be written as $v = v_0 + \lambda v_{1+} \lambda^2 v_2 + ...$ where λ will be set to 1 at the end of the problem. In this problem I wish you to find the functions v_n by solving the appropriate differential equations.

   a. Show that $v_0 = v_i$ where v_i is the initial velocity
   b. Show that $v_1 = \frac{(v_T - v_i)\, t}{\tau}$
   c. Show that $v_2 = -\frac{(v_T - v_i)\, t^2}{2\tau^2}$

2) We now have two different expansion for *v*(*t*), a short time one and a low velocity one. In this problem we wish to explore which approximation is better under what circumstances. To see this use Mathematica (or some other program) to plot the exact solution, the small velocity expansion (up to second order) and the small time expansion (up to second order) for time from zero to 2τ and for the initial conditions

   a) $v_i = \frac{v_T}{5}$.
   b) $v_i = 2v_T$

Does either approximation work better and if so in where? Why does this behavior make sense?

Next, I would like you to study a fictional world in which air resistance on an object is quadric in the velocity and directed opposite the direction of motion

3) In this problem you will derive the equation of motion for this situation.

a) Show that the equation of motion for one dimension motion (with down being defined as the positive direction) including the effects of gravity and air resistance is $m\frac{dv}{dt}=mg-\beta v^2$

b) Use dimensional analysis to find the characteristic time scale τ and the characteristic velocity scale v_t in terms of the parameters of the problem

c) Show that the equation of motion can be written as

$$\frac{d\left(\frac{v}{v_t}\right)}{dt}=\left(\frac{1}{\tau}\right)\left(1-\left(\frac{v}{v_t}\right)^2\right)$$

4) In this problem you should develop a perturbatve expansion for the preceding problem valid for low velocities. This expansion will in essence be an expansion in the strength of the force of air resistance. Accordingly we can rewrite the equation of motion as

$$\frac{d\left(\frac{v}{v_t}\right)}{dt}=\left(\frac{1}{\tau}\right)\left(1-\lambda\left(\frac{v}{v_t}\right)^2\right)$$

and express the velocity (or equivalently the ration of the velocity to the characteristic velocity) as a Taylor series in λ

$$\left(\frac{v}{v_t}\right)=\left(\frac{v}{v_t}\right)_0+\lambda\left(\frac{v}{v_t}\right)_1+\lambda^2\left(\frac{v}{v_t}\right)_2+\lambda^3\left(\frac{v}{v_t}\right)_3+\ldots$$

a) By equating coefficients of λ, show that the equations for the various terms in this expansion up to second order are given by

$$\frac{d\left(\frac{v}{v_t}\right)_0}{d\,t}=\left(\frac{1}{\tau}\right)$$

$$\frac{d\left(\frac{v}{v_t}\right)_1}{dt}=-\left(\frac{1}{\tau}\right)\left(\frac{v}{v_t}\right)_0^2$$

$$\frac{d\left(\frac{v}{v_t}\right)_2}{dt} = -2\left(\frac{1}{\tau}\right)\left(\frac{v}{v_t}\right)_0\left(\frac{v}{v_t}\right)_1$$

b) Solve the differential equations in part a) subject to the boundary conditions that at time t= 0,

$$\left(\frac{v}{v_t}\right)_0 = \left(\frac{v_i}{v_t}\right) \quad \left(\frac{v}{v_t}\right)_1 = \left(\frac{v}{v_t}\right)_2 = 0$$

and explain why these are the correct boundary conditions to use.