$\boxed{\delta\text{-function}}$ review: (1) $\int_{x=a+\varepsilon}^{a-\varepsilon} \delta(x-a) = 1$; (2) $\delta(x-a) = 0$ if $x \neq a$

and (3) $\int \phi(x)\,\delta(x-a)\,dx = \phi(a)$ if $a$ in region of $\int$...

# Green functions

$\boxed{\text{Sec. 12 of chapter } \boxed{8}}$: A brief introduction to Green functions here

— used for solving ODE (or PDE: chapter 13, best sec 8)

— illustrated by examples (ask)

(1) $y'' + \omega^2 y = f(t)$ e.g. describes oscillations of mass suspended by spring (or circuit with small resistance)

given (forcing) function

with initial condition (ask how many needed): $y_0 = y_0' = 0$

— Use Dirac $\delta$-function to re-write $f(t)$ as $\int_0^\infty f(t')\,\delta(t'-t)\,dt'$ ($\int$ over dummy variable)

[ general formula: $\int_a^b \phi(t)\,\delta(t-t_0)\,dt = \begin{cases} \phi(t_0), & a < t_0 < b \\ 0, & \text{otherwise} \end{cases}$

$\therefore$ $\delta$-function used to "sample" $\phi$ at $t_0$

finite $\delta$-function force: $\int$ of which is momentum imparted

— i.e., think of force $f(t)$ as (limiting case of) a whole sequence of impulses. If you wish, air pressure — e.g. of force — is on molecular level due to per unit area very large number of impacts of individual molecules, i.e., it's all "discrete" at fundamental level! ]

say most of it

skip

— suppose we have solved for $f(t)$ being $\delta(t'-t)$ (see later)

i.e., response of system to unit impulse at $t'$ (know) (since function/force is zero if $t \neq t'$): call it $G(t, t')$ ... need 2 "variables", since it's solution to

$$\frac{d^2}{dt^2} G(t, t') + \omega^2 G(t, t') = \delta(t'-t)$$

(not $dt'$)

[ if you prefer its function of $t$ (as always), but with label of $t'$, i.e., when was impulse... ]

Since $f(t)$ is adding-up many, many impulses, so is $\boxed{\text{solution}}$ to original ODE obtained by summing responses, $G$, to impulses, i.e., by direct substitution, show that

$$y(t) = \int G(t,t')\, f(t')\, dt' \quad (\text{only function of } t)$$

does it:

$$y'' + \omega^2 y = \left(d^2/dt^2 + \omega^2\right) \overbrace{\int G(t,t')\, f(t')\, dt'}^{y}$$

hits here only

$$= \int \underbrace{\left(d^2/dt^2\, G + \omega^2 G\right)}_{\delta(t'-t)} f(t')\, dt' = f(t)$$

— $\boxed{G}$ is called Green function (again, response of system to unit impulse at $t = t'$)

← using Laplace transform in prob. 1

— solve for $G$ with $G = 0$, $dG/dt = 0$ at $t = 0$ (same initial condition as on $y$):

but elem. on next page

$$G(t,t') = \begin{cases} 0 & 0 < t < t' \\ \frac{1}{\omega} \sin\omega\,(t - t') & 0 < t' < t \end{cases}$$

given

(only interested in $t \geq 0$)

... so that solution for given forcing function is

$t$ ← instead of $\infty$ since $G = 0$ for $t' > t$

$$y(t) = \int_0^{t} \frac{1}{\omega} \sin\omega\,(t - t')\, f(t')\, dt'$$

$\int$ over here

... similarly for other ODEs

## Solving for $\boxed{\text{Green's function}}$ for ODE: initial conditions

$y'' + \omega^2 y = f(t)$, subject to $y_0 = y_0' = 0$

using $\boxed{\text{elementary}}$ methods: we have

$$\left(d^2/dt^2 + \omega^2\right) G(t, t') = \delta(t' - t) \quad \ldots (1)$$

think of it "fixed" (& > 0); $\leftarrow$ varying

with $G(0, t') = 0$ and $dG/dt\,(t, t')\big|_{t=0} = 0$

— First note that for (any) $t \neq t'$, we have to solve

$$d^2/dt^2\, G(t, t') = -\omega^2 G(t, t') \quad (t \neq t') \quad \ldots (2)$$

... so that it's reasonable to try linear combinations of $\sin\omega t$ and $\cos\omega t$ [i.e, solutions to Eq. (2)] as solutions to Eq. (1) as well

... but these combinations could be different for $t < t'$ vs. $t > t'$, since after all, there is an impulse in-between (i.e., at $t = t'$)

— $\boxed{\text{idea}}$: use initial conditions (at $t=0$) & "patching" two solutions at $t = t'$

— Consider $\boxed{0 \leq t < t'}$ [again, we are interested in $t \geq 0$ (i.e, subsequent times)]: assume

$$G(t, t') = A \sin\omega t + B \cos\omega t \quad (0 \leq t < t')$$

$A$, $B$ could depend on $t'$

Now use initial conditions:

$G(0, t') = 0 \Rightarrow B = 0$

$dG/dt\,(t, t')\big|_{t=0} = +A\omega \cos\omega t\big|_{t=0} = 0 \Rightarrow A = 0$ (since $B = 0$)

so that $G(t, t') = \boxed{0}$ for $\boxed{t < t'}$

– Onto $\boxed{t > t'}$, when $G(t,t') = C\sin\omega t + \underbrace{D\cos\omega t}$

... but $G(t,t')$ is expected to be continuous
since respone can't "jump" even if there's impulse
at $t = t'$, i.e., $G(t'+\epsilon, t') = G(t'-\epsilon, t')$, with $\epsilon \to 0$

so that $\underbrace{C\sin\omega t' + D\cos\omega t'}_{\text{use } G \text{ for } t > t'} = \underbrace{0}_{G \text{ for } t < t'}$

$$\Rightarrow \quad D = -C\,\frac{\sin\omega t'}{\cos\omega t'}$$

so that $G(t,t') = C\sin\omega(t-t')$ (again, for $t > t'$)

However, we then see that derivative of $G$ is not continuous at $t = t'$: $\left.\frac{dG(t,t')}{dt}\right|_{t'-\epsilon} = 0$

$\boxed{\text{vs.}}$ $\left.dG/dt\,(t,t')\right|_{t'+\epsilon} = \left.C\omega\cos(t-t')\right|_{t'+\epsilon}$

$= C\omega$ (as $\epsilon \to 0$)

... which is as expected, since $\int_{t'-\epsilon}^{t'+\epsilon}$ of Eq.(1) gives (using $\int \frac{d^2G}{dt}\,dt = dG/dt$):

$$\left(\left.\frac{dG}{dt}\right|_{t=t'+\epsilon} - \left.\frac{dG}{dt}\right|_{t'=t'-\epsilon}\right) + \left(\omega^2\int_{t'-\epsilon}^{t'+\epsilon} G(t,t')\,dt\right) = \int_{t'-\epsilon}^{t'+\epsilon} \delta(t-t')$$

since $G$ is finite at $t = t'$, this is 0

i.e., $C\omega - 0 = 1 \Rightarrow C = 1/\omega$

so that $G(t,t') = \boxed{\frac{1}{\omega}\sin\omega(t-t')}$ $\boxed{t > t'}$