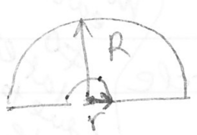
"Summary" of $\boxed{PV}$

— $\boxed{\text{Naively}}$ (using $\cos x/x$ is odd), we get $\int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx = 0$

— 2nd thought / step-back / re-think: $\cos x/x \to \infty$ near $x = 0$ $\Rightarrow$ $\int$ not well-defined! (mathematically!) (middle of integration region)

— As part of contour integral in e.g. $\boxed{4}$, we obtained (strictly speaking):



$$\lim_{R\to\infty} \lim_{r\to 0} \int_{-R}^{-r} + \int_{r}^{R} \frac{\cos x}{x} = 0 \quad \ldots \text{ but (again)}$$

r cancels (so does R)

but that's safe any-way

$\cos \ldots \to 0$ $R \to \infty$

cut-out region near origin

this is not quite $\int_{-\infty}^{\infty}$ ... so given different name (PV)

(although "looks like" it!)

— Analogy with $\int_0^5 \frac{dx}{x-3} = \ln|x-3|\Big|_0^5 = \ln 2/3$

(naively all OK)

... but upon more careful inspection: integrand $\to \infty$ for x near 3 (in middle of integration region) $\Rightarrow \int$ not well-defined

... can do $\int_0^{3-r} + \int_{3+r}^{5} \ldots$ (r divergence cancels) $= \ln 2/3$

cut-out ...

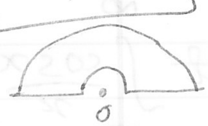
... but (again) this is PV of $\int \ldots$

↗ pole subtlety

$r \to 0$, $R \to \infty$

↳ no such issue

After showing, (explicitly in e.g. $\boxed{4}$) that $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = i\pi$



using contour going around / avoiding pole in e^{iz}/z at $z = 0$, try to "generalize", i.e., as $r \to 0$, above contour

→ real line ($-R$ to R) + semicircle, i.e., "original" C (as in e.g. 2, 3), but which passes straight thru' pole

suppose $\boxed{\text{start}}$ with original C

$\Rightarrow$ "suggests" $\oint_{C\ (\text{thru' pole})} = i\pi$ also

since $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx + \cancel{\int R \text{ semicircle}}^{\,0}$

$i\pi$ as above

but not $\int$, since that's 0 earlier, ≠0 later (due to pole on boundary)

i.e. $\oint_{C\ \text{thru' pole}}$

$= 2\pi i \, \frac{1}{2} \underbrace{R(0)}$

$= \lim z \to 0$ of $z \cdot \frac{e^{iz}}{z} = 1$

... $\boxed{\text{general}}$ rule (proof in problem 21: informal HW)

$\oint_C f(z) = (2\pi i)\Big[$ sum of residues $\boxed{\text{fully}}$ inside (clear)

$+ \frac{1}{2}$ sum of residues at/on boundary $\Big]$ at simple poles

since 1 if inside } so ½ when "in middle"
0 if outside }

[$\int_0^5 \frac{dx}{x-3} \overset{\text{naively}}{=} \ln|x-3|\Big|_0^5 = \ln 2/3$

then $\textcircled{2}^{nd}$ thought / re-think / step back: integrand $\to \infty$ "on the way"!!

→ cut-out small symmetric interval about $x = 3$:

$$\int_0^{3-r} \frac{dx}{x-3} = \ln|x-3|\Big|_0^{3-r} = \ln r - \ln 3$$

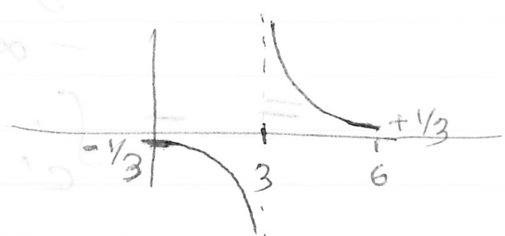
$$\int_{3+r}^{5} \frac{dx}{x-3} = \ln 2 - \ln r$$

as $r \to 0$

∴ sum $= \ln 2/3$, independent of r (again, 2 ∞ cancel)

PV $\int_0^5 \frac{dx}{x-3} = \ln 2/3$

$\ln r$ & $-\ln r$



∞ area above x axis and below cancel

... just like in contour integral above, $\int_{-\infty}^{-r}$ and $\int_{+r}^{\infty}$, then $r \to 0$ ... it is finding PV in above e.g... so (again) 0 is $\boxed{\text{PV}}$ of $\int_{-\infty}^{\infty} \frac{\cos x}{x} dx$ ..

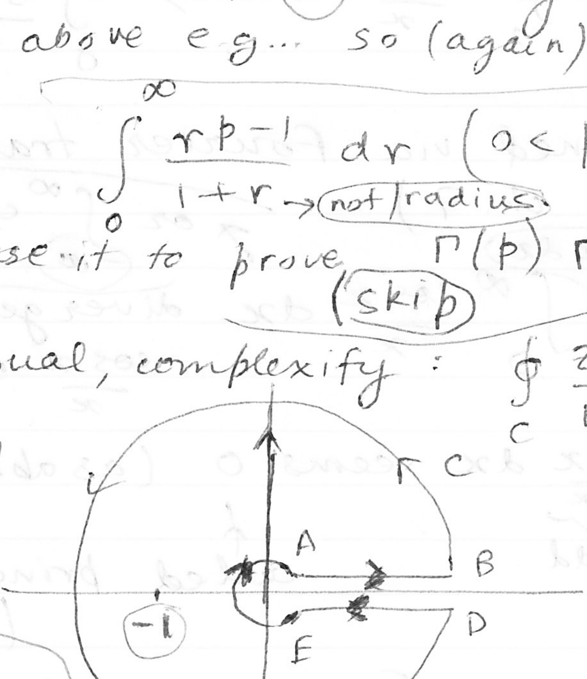
E.g. $\boxed{5}$ $\int_0^\infty \frac{r^{p-1}}{1+r} dr \quad (0 < p < 1)$ so that it's $r^{<0}$ i.e., $\to \infty$ as $r \to 0$

$1 + r \to$ not / radius

[use it to prove $\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin \pi p}$] (skip)

... as usual, complexify: $\oint_C \frac{z^{p-1}}{1+z} dz \quad (0 < p < 1)$

around



(again, looks like pole at $\boxed{z = 0}$ due to $z^{<0}$ ...)

so, go "around" it ...

$\int_{-\infty}^{0}$ not related to $\int_0^\infty$ ... so can't use earlier C

e.g., $p = 1/2$ is $1/\sqrt{z}$

... but wait: meaning of z^{p-1} (<0) fraction!, i.e., not single-valued (like $\ln z$)

[need to explain: just above positive real axis, i.e., $x>0, y=+\epsilon$ we have $\theta=0$ ... but just below, it's $\theta=2\pi$ (different branch)
$\Rightarrow$ +ve real axis is branch cut ... but −ve real axis "kosher", i.e., both $y=\pm\epsilon$ for $x<0$ have $\theta\simeq\pi$ ...] → that's why contour

($p=1/2$) two square roots of any z: say, at $\theta=\pi/4$, we have $z=re^{i\pi/4}$; $z^{-1/2}=r^{-1/2}e^{-i\pi/8}$

... but if $\theta=\pi/4 \overset{\theta}{+} 2\pi$ ("back to" starting point after following circle), then $z=re^{i(\pi/4+2\pi)}$

$\Rightarrow \frac{1}{\sqrt{z}} = r^{-1/2}e^{-i(\pi/8+\pi)} = - r^{-1/2}e^{-i\pi/8}$

— same is true for any other starting point (other θ) $(r\neq 0)$: $\frac{1}{\sqrt{z}}$ or z^{1-p} comes back to different value (branch) when $\theta\to\theta+2\pi$ (return to starting point)

— so, for z^{p-1} to be single-valued, choose 1 branch of z^{p-1}, i.e., decide some 2π interval for θ, say $\boxed{0 \text{ to } 2\pi}$

i.e., imagine artificial barrier or branch cut along positive x-axis ← not to be crossed!

- a point which we cannot encircle [even with (very) small circle] w/o crossing branch cut (i.e., going to another branch) is called branch point, e.g., origin here

not needed ?

— $\theta=0$ along AB (just above x-axis); follow C around to $\boxed{DE}$: θ increases by 2π, i.e., $\boxed{\theta=2\pi}$ is on lower side of x-axis

— contour C maintains $\boxed{\theta}$ between 0 and 2π $\Rightarrow$ → stay on 1 branch

z^{p-1} is single valued ...

... and $z^{p-1}/(1+z)$ is $\boxed{\text{analytic}}$ inside it, except for pole at $z=-1=e^{i\pi}$, with residue $\lim_{z\to -1}(1+z)\frac{z^{p-1}}{1+z}$

(again, go around $z=0$)

$= (e^{i\pi})^{p-1}$

$= \underbrace{e^{-i\pi}}_{-1} e^{i\pi p}$

... so that (residue theorem) $\oint \frac{z^{p-1}}{1+z}dz$

$= -2\pi i\, e^{i\pi p}$ $(0<p<1)$

— along 2 $\boxed{\text{circles}}$ $z=re^{i\theta}$ so that we get

$$\int \frac{r^{p-1}e^{i(p-1)\theta}}{1+re^{i\theta}} \underbrace{rie^{i\theta}d\theta}_{dz} = i\int \frac{r^p e^{ip\theta}}{1+re^{i\theta}}d\theta$$

$$\int \underbrace{\frac{r^p e^{ip\theta}}{}}_{\approx 1} d\theta \underset{r\to 0}{\rightarrow} \int \frac{0}{1} e^{ip\theta} d\theta$$

– remarkably, $\int \to 0$ for both $r \to 0$ & $r \to \infty$

i.e., $\int$ along circular part $\to 0$ as inner circle shrinks, outer one expands

$$\int \frac{r^p e^{ip\theta}}{\approx re^{i\theta}} d\theta \underset{r\to\infty}{\longrightarrow} \int \frac{1}{\infty^{\underbrace{(1-p)}_{<0}}} e^{i(p-1)\theta} d\theta$$

$\Rightarrow$ "left" with $\int$ on AB (0 to ∞) and DE (∞ to 0) (along x-axis)

AB: $\theta = 0$ so $z = re^{i\cdot 0} = r$ i.e., $\underbrace{\int_{r\to 0}^{\infty} \frac{r^{p-1}}{1+r} dr}_{\text{wanted}}$

... vs. $\boxed{DE}$: $\theta = 2\pi$, $z = re^{2\pi i}$, giving

$$\int_{r\to\infty}^{0} \frac{(re^{2\pi i})^{p-1}}{1+re^{2\pi i}} e^{2\pi i} dr = -\int_0^{\infty} \frac{r^{p-1} e^{2\pi i p}}{1+r} dr$$

wanted

$\Rightarrow$ adding AB & DE gives $\left(\int \frac{r^{p-1}}{1+r} dr\right)(1-e^{2\pi i p})$

$= -2\pi i e^{i\pi p}$

residue

so that $\int_0^{\infty} dr \frac{r^{p-1}}{1+r} = \frac{-2\pi i e^{i\pi p}}{1-e^{2\pi i p}}$

$$= \frac{\pi\, 2i}{e^{i\pi p} - e^{-i\pi p}} = \frac{\pi}{\sin \pi p}$$