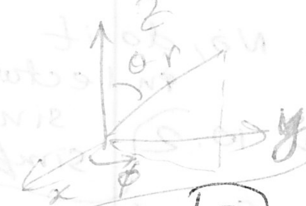


(ask:) motivates associated Legendre functions from equation for $\cos\theta$ (-1 to 1 etc.) — maybe, let this "pop-out"

sec. 7 Steady-state temperature in a sphere

no time-dependence

— find steady-state temperature $\boxed{\text{in}}$side sphere of radius a if surface of upper half held at 100° vs. lower half at 0° — or some other BC

— temperature u satisfies Laplace's eqn. (like for rectangular plate); choose to write it in spherical coordinates to match BC:

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \quad \cdots ①$$

– Try: $u = R(r)\,\Theta(\theta)\,\Phi(\phi)$; plug; $\times \underbrace{\frac{r^2}{R\Theta\Phi}}_{u}$ to get:

do it slowly since last time

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0 \quad \cdots ②$$

$\left(-\frac{m^2}{\sin^2\theta}\right)$ (see below)

$\times \sin^2\theta$ so that last term is function of ϕ only and other terms don't contain ϕ $\Rightarrow$

i.e., $\Theta\Phi\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)$ ...

set $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2$, i.e., $\Phi = \left\{\begin{matrix}\sin m\phi \\ \cos m\phi\end{matrix}\right\}$ (ask)

[must be $-m^2$, with m integer so that Φ is periodic function of ϕ ... like for θ in cylindrical case...]

$\Rightarrow$ ② can be written as above ... (i.e., no ϕ on LHS & RHS $= -\frac{m^2}{\sin^2\theta}$) separate further (1st term is function of r only, other 2 contain θ) $\Rightarrow$ (ask)

2nd separation constant

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = k \quad \& \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + k\Theta = 0$$

HW ? ← No, do it in lecture, since simple

problem 10.2

... Θ eqn. is for associated Legendre functions

$\boxed{\text{if}}$ $k = \ell(\ell+1)$ ... so far, k & ℓ arbitrary...

but ℓ must be integer in order for solution of (simple) Legendre eqn. be finite / series convergent at x ($=\cos\theta$ here) $= \pm 1$ ($\theta = 0$ or π) : same true for eqn. for associated Legendre functions ... ⇒ which are derivatives of P_ℓ

k must be $\ell(\ell+1)$ (integer) here ... so that

$$\Theta = P_\ell^m(\cos\theta)$$

Put $k = \ell(\ell+1)$ in r eqn. ⇒ $R(r) = \begin{cases} r^\ell \\ r^{-\ell-1} \end{cases}$

ask

discard because blows up at origin which we are interested in

if interested in outside sphere only, use $r^{-\ell-1}$ only (because r^ℓ blows up at infinity)

— So, basis functions are

$$u = r^\ell P_\ell^m(\cos\theta) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}$$

[$P_\ell^m(\cos\theta) \times \sin m\phi$ (or $\cos$) $= Y_\ell^m(\theta,\phi)$ are called spherical harmonics]

— If temperature at $r=a$ given as function of θ & ϕ, then solution inside is double series (sum ℓ, m) ($-\ell \le m \le \ell$)

— For $\boxed{\text{given}}$ case, temperature independent of ϕ ⇒ $m=0$ and only $\cos m\phi = 1$ ⇒ $\sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos\theta)\, c_\ell = u$

... as usual, determine c_ℓ using BC at $r=a$

e.g. if $\Theta(\theta) = \cos\theta$, then $y = x$

Equation for $\Theta(\theta)$:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + k\Theta = 0$$

change variables $x = \cos\theta$; $\Theta(\theta) = y(x)$

$y' = dy/dx$

$$\Rightarrow \frac{d}{d\theta}\underbrace{y(x)}_{\Theta(\theta)} = \frac{d\,y(\cos\theta)}{d\cos\theta}\,\frac{d\cos\theta}{d\theta}$$

$$= y'\Big|_{x=\cos\theta}(-\sin\theta)$$

do it 1st $\frac{d\Theta}{d\theta}$

i.e, $\frac{d}{d\theta} = \frac{d}{dx} \times (-\sin\theta)$

$$\underbrace{\frac{1}{\sqrt{1-x^2}}}_{1/\sin\theta}\underbrace{(-\sin\theta)}_{\sqrt{1-x^2}}\frac{d}{dx}\left(\sqrt{1-x^2}\,\underbrace{\left(-\sqrt{1-x^2}\right)}_{-\sin\theta}\frac{dy}{dx}\right)$$

$$- \frac{m^2}{\sin^2\theta\}(1-x^2)} + ky = 0$$

$$+\frac{d}{dx}\left[(1-x^2)\,y'\right] + \left[-\frac{m^2}{(1-x^2)} + k\right]y = 0$$

i.e, $(1-x^2)y'' - 2x\,y' + \left[k - \frac{m^2}{(1-x^2)}\right]y = 0$

upper half

i.e., $\sum_{l=0}^{\infty} \overset{\psi\, r}{a^l} c_l P_l(\cos\theta) = \begin{cases} 100 & 0<\theta<\pi/2 \\ & 0<\cos\theta(\equiv x)<1 \\ 0 \text{ for } \pi/2<\theta<\pi & -1<x<0 \end{cases}$

or $\sum_{l=0}^{\infty} a^l c_l P_l(x) = 100\, f(x)$

not coordinate x but just $\cos\theta$

$f(x) = \begin{cases} 0 & \text{for } -1<x<0 \\ +1 & \text{for } 0<x<1 \end{cases}$

above $f(x)$ (from Ex 9.1) $= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + \cdots$

so that $u = 100\left[\frac{1}{2}\underbrace{P_0(\cos\theta)}_{=1} + \frac{3}{4}\frac{r}{a} P_1(\cos\theta) + \cdots\right]$

Variations

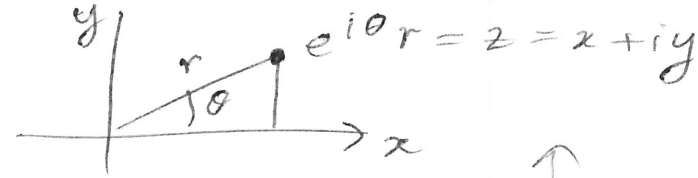
- °C vs °F (related by multiplying by & adding constant): use that $u + C$ and Cu are also solutions of Laplace's equation (or heat flow) $\quad °C = 5/9(°F - 32)$

- temperature at $\theta = \pi/2$ ($\cos\theta = 0$), i.e., equatorial plane – as given by Legendre series – is ½ of top & bottom surface because converges to midpoint of jump ($r = a$)

  ... so solving problem of temperature in a hemisphere given that on curved surface & equatorial plane: "imagine" lower hemisphere in place at temperature such that get desired average on equatorial plane, e.g., if 100° on curved surface and 0° on equatorial plane, extend $f(x)$ on $(-1, 0)$ to make it odd i.e., $f(x) = \begin{cases} 1 & 0<x<1 \\ -1 & -1<x<0 \end{cases}$ vs 0 above

  (i.e., "lower" hemisphere surface at −100°)

  so that midpoint of jump at $x = 0$ is 0



Chapter $\boxed{14}$ of Boas (Functions of a complex variable)

$\boxed{\text{sec 1}}$ Introduction : $z = x + iy$ in complex (xy) plane ; elementary functions of z

(roots, log, trigonometric) ... in chapter $\boxed{2}$

– $\boxed{\text{Now}}$, calculus of functions of z (differentiation; integration; power series) ↳ value is complex

– motivation : convenient to use complex expressions (even if not a must), e.g., Fourier sine/cosine series is complex in short-hand ∴ simplify many calculations + lead to better understanding of problem, hence more efficient method of solution... i.e., start real, introduce complex later

i.e., maybe start with complex

– e.g., $f(z) = z^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2ixy}_{v(x,y)}$ make mistake: e.g., $x^2 + y^2$

∴ just like complex number (z) equivalent to pair of real numbers (x, y), so is function of z ... to pair of real functions : u, v (of real variables x, y)

i.e., in general $f(z) = f(x + iy) = u(x,y) + i\,v(x,y)$

– functions are usually single-valued, i.e., 1 value (complex) for each z ... but what about $\ln z = \ln|z| + i(\theta + 2n\pi)$ where $\tan\theta = y/x$ (i.e., $z = \sqrt{x^2+y^2}\, e^{i\theta}$) ?

⇒ each z, $\ln z$ has ∞ set of values (labelled by n)

... but $\boxed{\text{if}}$ θ is allowed only 2π range, then $\ln z$ is single-valued (called branch of $\ln z$) for $\ln z$ (similarly $\sqrt{z}$, $\tan^{-1} z$)

i.e., there is a collection of branches {(multiple valued function) but discuss 1 branch at a time (again, single-valued function)

$\boxed{\text{Sec. 2}}$ $\boxed{\text{Analytic functions}}$

— Derivative of $f(z)$: $f'(z) = \dfrac{df}{dz} = \lim\limits_{\Delta z \to 0} \dfrac{\Delta f}{\Delta z}$

where $\Delta f = f(z + \Delta z) - f(z)$, with $\Delta z = \Delta x + i \Delta y$

— $\boxed{\text{Analytic}}$ (or regular, harmonic, monogenic) in $\boxed{\text{region}}$ of complex plane if it has a unique derivative at every point of the region

2D (i.e., isolated points and curves are not regions)

[analytic at point $z = a$ $\Rightarrow$ derivative at every point inside some "small" circle about $z = a$]

— Onto meaning of derivative for complex case: 

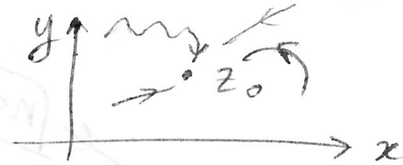
— real case 1st: derivative can be discontinuous (i.e., 2 values at point x_0: left approach vs. right) $\Rightarrow$ if we say $f(x)$ has derivative at $x = x_0$, we mean 2 values are equal...

... but for complex case, ∞ number of ways to approach z_0 $\Rightarrow$ $f'(z)$ has same value no matter how (among ∞ ways) we approach z_0 ... stringent requirement!

e.g., $f(z) = z^2 \Rightarrow f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{(z + \Delta z)^2 - z^2}{\Delta z}$

$= \lim (2z + \Delta z) = 2z$

... independent ...



$f(z)$ plotted on 3rd axis out of page

... similarly, $d/dz\,(z^n) = n z^{n-1}$ (complex)

... could have (naively) guessed, based on definition of derivative being same form as for function of a real variable; (similarly, many other formulae can be proved)

sum, product, quotient rule + derivatives of rational functions, other elementary functions; also chain rule [i.e., if $f = f(g)$ and $g = g(z)$, then $df/dz = df/dg \; dg/dz$]

... so, what's "new" here? (since results same as for real case)

— that's true for functions of z which have derivatives ... but then we didn't "exploit" ∞ number of ways to approach z_0 → maybe more interesting ones?

— e.g. of non-analytic functions: $d/dz\,|z|^2$

($\text{cf. } d/dx\,|x|^2 = d/dx\, x^2 = 2x$, i.e., exists!)

i.e., same form of function of real variable

try $\lim_{\Delta z \to 0} \dfrac{|z+\Delta z|^2 - |z|^2}{\Delta z}$ } always real

$\Delta z = \Delta x + i\Delta y$ ⇒ if we approach along horizontal line, $\Delta y = 0 \Rightarrow \Delta z = \Delta x$ (purely real) ... but along vertical line, $\Delta x = 0 \Rightarrow \Delta z = i\Delta y$ (purely imaginary) and along other direction, Δz is complex

$\Rightarrow \lim_{\Delta z \to 0} \dfrac{\Delta f}{\Delta z}$ } real / } general complex ⇒ clearly different values for different directions of approach

$\boxed{\text{Cauchy-Riemann}}$ conditions for telling whether analytic or not: If $f(z)$ is analytic, then in that