Proof of 1^{st} equality of Eq. 7.8 from chapter 14 of Boas (problem 7.42 from chapter 14)

$\boxed{(a)}$ Residue of $F(z) = f'(z)/f(z)$ at zero of $f(z)$ of order n (at $z = a$): we have expansion of f around $z = a$ of the form

$$f(z) = a_n (z-a)^{\underset{\text{order of zero}}{n}} + a_{n+1} (z-a)^{n+1} \ldots$$

[no negative powers of $(z-a)$, since it's not a pole of $f(z)$]

Thus, $f'(z)$ [around $z = a$] $\overset{\text{differentiating above series term-by-term}}{=}$ $n\, a_n (z-a)^{n-1} + (n+1)\, a_{n+1} \times (z-a)^n + \ldots$

so that we get

$$F(z)\,[\text{around } z = a] \overset{\text{plugging in above series for } f \text{ and } f'}{=} \frac{[n\, a_n (z-a)^{n-1} + (n+1)\, a_{n+1} (z-a)^n + \ldots]}{[a_n (z-a)^n + a_{n+1} (z-a)^{n+1} + \ldots]}$$

$$\overset{\text{dividing each term in numerator \& denominator by } (z-a)^n}{=} \frac{[n\, a/(z-a) + (n+1)\, a_{n+1} + \ldots]}{[\underbrace{a_n}_{\text{as leading term}} + a_{n+1} (z-a)^1 + \ldots]}$$

Due to $1/(z-a)$ in numerator (vs. constant in denominator), there is pole for $F(z)$ at $z = a$ (which was expected anyway from start, since F has f in denominator)

So, residue of F (at $z = a$) $= \lim_{z \to a} (z-a) F(z)$ ← plug above series

$= [n\, a_n + 0 + \ldots]/[a_n + 0 + \ldots]$

$= \boxed{n}$

$\boxed{(b)}$ Similarly, residue of $\underline{F}$ at $\underline{\text{pole}}$ of $f(z)$ at $\underline{z=b}$ of order $\underline{p}$: we have the $\underline{\text{Laurent}}$ series:

$$f(z)\,[\text{around } z=b] = a_0 + a_1(z-b) + a_2(z-b)^2 + \dots$$
$$+ \frac{b_1}{z-b} + \dots + \frac{b_p}{(z-b)^p} \quad \left(\underline{\text{no}}\ b_{p+1} \text{ onwards}\right)$$

so that
$$f'(z)\,[\text{around } z=b] = a_1 + 2a_2(z-b)^1 + \dots$$
$$- \frac{b_1}{(z-b)^2} + \dots \frac{(-p)\,b_p}{(z-b)^{p+1}}$$

Thus $F(z)\,[\text{around } z=b] =$
$$\frac{\left[a_1 + 2a_2(z-b) + \dots - b_1/(z-b)^2 + \dots \frac{(-p)\,b_p}{(z-b)^{p+1}}\right]}{\left[a_0 + a_1(z-b) + \dots + b_1/(z-b) + \dots + b_p/(z-b)^p\right]}$$

$\underline{\text{multiply}}$ numerator & denominator by $(z-b)^p$ (and change order of terms)

$$= \frac{\left[-\frac{p\,b_p}{(z-b)} + \dots - b_1(z-b)^{p-2} + a_1(z-b)^p + 2a_2(z-b)^{p+1} + \dots\right]}{\left[b_p + \dots b_1(z-b)^{p-1} + a_0(z-b)^p + a_1(z-b)^{p+1} + \dots\right]}$$

Clearly, there is a pole in $\underline{F}(z)$ at $z=b$, with $\underline{\text{residue}} = \lim_{z\to b} (z-b)\,F(z)$

$$= [-p\,b_p + 0 + \dots]/[b_p + 0 + \dots]$$

$$= \boxed{-p}$$

Summing over all zeroes & poles of f, we get 1st equality of Eq. 7.8