PHYS 373 (Fall 2021): Mathematical Methods for Physics II

Summary of topics/formulae covered in this course

Chapter 7 of Boas (Fourier Analysis)

Fourier *series*

1. General Fourier series: a function $f(x)$ with period $2l$ can be expanded as

$$f(x) = \frac{a_0}{2} + \sum_1^\infty \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right) \tag{1}$$

$$= \sum_{-\infty}^{\infty} c_n e^{in\pi x/l} \tag{2}$$

where the Fourier coefficients are given by

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \tag{3}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \tag{4}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx \tag{5}$$

2. Special Fourier series: we have

$$\text{if } f(x) \text{ is odd,} \quad \begin{cases} b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \\ a_n = 0 \end{cases} \tag{6}$$

$$\text{if } f(x) \text{ is even,} \quad \begin{cases} a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx \\ b_n = 0 \end{cases} \tag{7}$$

3. Paserval's theorem for Fourier series:

$$\text{The average of } |f(x)|^2 \text{ (over a period)} = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + +\frac{1}{2}\sum_1^\infty b_n^2 \tag{8}$$

$$= \sum_{-\infty}^{\infty} |c_n|^2 \tag{9}$$

Fourier ***Transform***

1. General Fourier transform

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x} d\alpha \tag{10}$$

where

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \tag{11}$$

2. Special Fourier transform: for an odd function, we have

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\alpha) \sin \alpha x \, d\alpha \tag{12}$$

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \sin \alpha x \, dx \tag{13}$$

Similarly, for an even function:

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(\alpha) \cos \alpha x \, d\alpha \tag{14}$$

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(x) \cos \alpha x \, dx \tag{15}$$

$$\tag{16}$$

3. Paserval's theorem for Fourier transform:

$$\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} \frac{1}{2\pi} |f(x)|^2 dx \tag{17}$$

Chapter 8 of Boas (Ordinary Differential Equations)

Simple: 1st or 2nd order

1. Simply integrate both sides:

$$\begin{aligned} y' \text{ or } y'' &= f(x) \Rightarrow \\ y \text{ or } y' &= \int dx f(x) \end{aligned} \tag{18}$$

1st order

1. 1st order, linear, separable (integrate after bit of manipulation):

$$\begin{aligned} y' &= f(x) g(y) \Rightarrow \\ \int \frac{dy}{g(y)} &= \int dx f(x) \end{aligned} \tag{19}$$

2. 1st order, linear, but *not* separable:

$$\begin{aligned} y' + P(x)y &= Q(x) \Rightarrow \\ y &= e^{-I} \int dx\, Q\, e^{I} + c\, e^{-I}, \text{ where} \\ I &= \int P\, dx \end{aligned} \tag{20}$$

2nd order

1. 2nd order, linear, with constant coefficients, *and* RHS zero:

$$\begin{aligned} a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y &= 0 \\ = a_2(D-a)(D-b)y, \quad \text{where } D \equiv \tfrac{d}{dx} &\Rightarrow \\ y &= c_1 e^{ax} + c_2 e^{bx}, \text{ for } a \neq b \end{aligned} \tag{21}$$

2. 2nd order, linear, with constant coefficients, *but* RHS *non*-zero:

$$\begin{aligned} a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y &= k e^{cx} \\ = a_2(D-a)(D-b)y, \quad \text{where } D \equiv \tfrac{d}{dx} &\Rightarrow \\ y &= y_p + \left(c_1 e^{ax} + c_2 e^{bx}\right), \text{ for } a \neq b, \text{ where} \\ y_p &= C\, e^{cx} \text{ for } a \neq c \neq b, \text{ with} \\ C &= \frac{k}{(c-a)(c-b)} \end{aligned} \tag{22}$$

3. Cauchy-Euler equation:

$$\begin{aligned} a_2 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_0 y &= f(x) \Rightarrow \\ a_2 \frac{d^2y}{dz^2} + (a_1 - a_2) \frac{dy}{dz} + a_0 y &= f\left(e^z\right), \text{ where} \\ x &= e^z \end{aligned} \tag{23}$$

which can be solved as above. Specifically, for $f = 0$, try $y \sim x^k$, solving for k.

4. "Conservation of energy"-type:

$$\begin{aligned} y'' + f(y) &= 0 \Rightarrow \\ \frac{1}{2} y'^{\,2} + \int f(y) dy &= \text{constant} \end{aligned} \tag{24}$$

which is separable.

Chapter 12 of Boas (Series Solutions of Differential Equations)

General method

1. Series method for solving (linear) ordinary differential equations (ODE): assume a solution of the form (with a's being coefficients *to be found*)

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{25}$$

giving

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \tag{26}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \tag{27}$$

Plug the above series into each term of the ODE. Find the *total* coefficient of each power of x on each side of ODE and equate them (again, for each power of x). This will give the higher a coefficients in terms of lower ones.

Legendre equation/polynomials

1. Legendre's equation:

$$\left(1 - x^2\right) y'' - 2xy' + l(l+1)y = 0 \tag{28}$$

has a solutions for each integer l (chosen to be non-negative) which is called the Legendre polynomial, $P_L(x)$ defined with

$$P_l(1) = 1 \tag{29}$$

For example,

$$P_0(x) = 1,\ P_1(x) = x,\ \ P_2(x) = \tfrac{1}{2}\left(3x^2 - 1\right),\ P_3(x) = \tfrac{1}{2}\left(5x^3 - 3x\right) ... \tag{30}$$

2. Rodrigues' formula for Legendre polynomials

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left(x^2 - 1\right)^l \tag{31}$$

3. Generating function for Legendre polynomials:

$$\Phi(x,h) = \left(1 - 2xh + h^2\right)^{-1/2},\ |h| < 1 \tag{32}$$

$$= \sum_{l=0}^{\infty} h^l P_l(x) \tag{33}$$

4. Recursion relations for Legendre polynomials:

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x), \tag{34}$$

$$xP_l'(x) - P_{l-1}'(x) = lP_l(x), \tag{35}$$

$$P_l'(x) - xP_{l-1}'(x) = lP_{l-1}(x), \tag{36}$$

$$\left(1-x^2\right)P_l'(x) = lP_{l-1}(x) - lxP_l(x), \tag{37}$$

$$(2l+1)P_l(x) = P_{l+1}'(x) - P_{l-1}'(x), \tag{38}$$

$$\left(1-x^2\right)P_{l-1}'(x) = lxP_{l-1}(x) - lP_l(x) \tag{39}$$

5. Orthogonality of Legendre polynomials:

$$\int_{-1}^{1} P_l(x)P_m(x)dx = 0, \text{ unless } l = m \tag{40}$$

6. Normalization of Legendre polynomials:

$$\int_{-1}^{1} \left[P_l(x)\right]^2 = \frac{2}{2l+1} \tag{41}$$

$$\tag{42}$$

7. A function defined over the interval $(-1,1)$ can be expanded in a Legendre series

$$f(x) = \sum_{l=0}^{\infty} c_l P_l(x) \tag{43}$$

$$\tag{44}$$

where

$$c_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_l(x)dx \tag{45}$$

Bessel equation/functions

1. Bessel equation

$$x^2y'' + xy' + \left(x^2 - p^2\right)y = 0 \tag{46}$$

has solutions (Bessel functions):

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)}\left(\frac{x}{2}\right)^{2n+p} \tag{47}$$

and

$$N_p(x) = \frac{\cos(\pi p)J_p(x) - J_{-p}(x)}{\sin \pi p} \tag{48}$$

2. Asymptotic values of Bessel functions:

$$J_0(0) = 1 \tag{49}$$
$$J_{n\neq 0}(0) = 0 \tag{50}$$
$$J_{n=0,1,2\ldots}(\infty) = 0 \tag{51}$$

3. Recursion relations for Bessel functions

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \tag{52}$$
$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x) \tag{53}$$
$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x) \tag{54}$$
$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x) \tag{55}$$
$$J_p'(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x) = \frac{p}{x} J_p(x) - J_{p+1}(x) \tag{56}$$

4. Orthogonality of Bessel functions:

$$\int_0^1 x J_p(\alpha x) J_p(\beta x) = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \frac{1}{2} J_{p+1}^2(\alpha) = \frac{1}{2} J_{p-1}^2(\alpha) = \frac{1}{2} J_p'^{\,2}(\alpha) & \text{if } \alpha = \beta \end{cases} \tag{57}$$

where α and β are zeroes of $J_p(x)$.

Chapter 13 of Boas (Partial Differential Equations)

Laplace's equation

1. Laplace equation in two-dimensional *rectangular/Cartesian* coordinates (for example, for steady-state temperature):

$$\frac{\partial^2}{\partial x^2} T(x,y) + \frac{\partial^2}{\partial y^2} T(x,y) = 0 \tag{58}$$

has basis functions (i.e., general solution is a suitable *combination* of these):

$$T(x,y) = \left\{ \begin{array}{c} e^{kx} \\ e^{-kx} \end{array} \right\} \left\{ \begin{array}{c} \sin ky \\ \cos ky \end{array} \right\} \text{ or } \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} \left\{ \begin{array}{c} e^{ky} \\ e^{-ky} \end{array} \right\} \tag{59}$$

2. Laplace equation in *spherical* coordinates

$$\frac{1}{r^2} \frac{\partial}{\partial} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial\phi^2} = 0 \tag{60}$$

has basis functions (where l is a non-negative integer, with $-l \leq m \leq +l$)

$$u = \left\{ \begin{array}{c} r^l \\ r^{-l-1} \end{array} \right\} P_l^m(\cos\theta) \left\{ \begin{array}{c} \sin m\phi \\ \cos m\phi \end{array} \right\} \tag{61}$$

where P_l^m are *associated* Legendre functions (which reduce to Legendre polynomials for $m = 0$)

Schroedinger equation

1. in one dimension for a *free* particle (i.e., no potential):

$$-\frac{h^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = ih\frac{\partial\Psi}{\partial t} \tag{62}$$

has basis functions

$$\Psi = \left\{ \begin{array}{c} \sin kx \\ \cos kx \end{array} \right\} e^{-iEt/h} \tag{63}$$

Wave equation

1. Wave equation in *circular* coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} \tag{64}$$

has basis functions:

$$z = \left\{ \begin{array}{c} J_n(Kr) \\ N_n(Kr) \end{array} \right\} \left\{ \begin{array}{c} \sin n\theta \\ \cos n\theta \end{array} \right\} \left\{ \begin{array}{c} \sin Kvt/a \\ \cos Kvt/a \end{array} \right\} \tag{65}$$

Chapter 14 of Boas (Functions of a Complex Variable)

Derivatives

1. Basics of complex-valued functions of complex variable

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \tag{66}$$

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \tag{67}$$

2. If $f(z)$ is analytic in a region (i.e., has a unique derivative at every point), then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{68}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{69}$$

(Cauchy-Reimman conditions) and it's converse: if $u(x,y)$ and $v(x,y)$ satisfy these conditions, then $f(z) = u + iv$ is analytic.

3. If $f(z)$ is analytic in a region R, then it has derivatives of all orders at points inside R and thus it can be expanded in a Taylor series about any point z_0 in R. This power series converges *in*side circle C about z_0 that extends to the nearest singularity point (i.e., C *just* touches the boundary of R).

4. If $f(z) = u + iv$ is analytic in a region, then u and v satisfy (two-dimensional) Laplace's equation in the region. And, conversely, *any* function u (or v) satisfying Laplace's equation is the real (or imaginary) part of an analytic function $f(z)$.

Integrals

1. Cauchy's theorem: if $f(z)$ is analytic inside and on a closed curve C, then

$$\int f(z)dz = 0, \text{ around } C \tag{70}$$

2. Cauchy's integral formula: if $f(z)$ is analytic inside and on a closed curve C, then

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}dz, \text{ around } C \tag{71}$$

where $z = a$ is a point *in*side C.

(Towards) Residue theorem

1. Laurent series: Let C_1 and C_2 be two circles with center at z_0. If $f(z)$ is an anlaytic function in the region R between $C_{1,2}$, then it can be expanded in a convergent series in R

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ... + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... \tag{72}$$

associated with which are the following definitions:

(i) If all the b's are zero, then $f(z)$ is analytic at $z = z_0$ (regular point);

(ii) If $b_n \neq 0$, but all the subsequent b's are zero, then $f(z)$ is said to have a *pole of order* n a $z = z_0$. If $n = 1$ here, then $f(z)$ has a simple pole at $z = z_0$;

(iii) If there are infinite number of b's which are different than zero, then $f(z)$ has an *essential* singularity at $z = z_0$;

(iv) The coefficient b_1 of $1/(z - z_0)$ is called the *residue* of $f(z)$ at $z = z_0$.

2. Residue theorem:

$$\int f(z)dz \text{ (around } C) = 2\pi i.\text{(sum of residues of } f(z)\text{inside } C) \tag{73}$$

where we go *counter*-clockwise around C.

3. Methods of finding residues of $f(z)$:

(A) coefficient b_1 in Laurent series about $z = z_0$;

(B) Simple pole:

$$R(z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{74}$$

and if $f(z) = g(z)/h(z)$, then

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \text{ if } \begin{cases} \text{if } g(z_0) = \text{ finite const.} \\ h(z_0) = 0,\ h'(z_0) \neq 0 \end{cases} \tag{75}$$

(C) Multiple poles: multiply $f(z)$ by $(z - z_0)^m$, where m is an integer $\geq n$ (order of pole), differentiate the result $(m-1)$ times, divide by $(m-1)!$, and evaluate the resulting expression at $z = z_0$.

4. Definite integrals using residue theorem:

(i) Change of variables;

(ii) If $P(x)$ and $Q(x)$ are polynomials with degree of $Q \geq$ degree of $P + 2$ and if Q has no real zeroes, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} = 2\pi i. \left(\text{sum of residues of } \frac{P(z)}{Q(z)} \text{ in upper half-plane}\right) \tag{76}$$

(iii) If $P(x)$ and $Q(x)$ are polynomials with degree of $Q \geq$ degree of $P + 1$ and if Q has no real zeroes, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{imx} = 2\pi i. \left(\text{sum of residues of } \frac{P(z)}{Q(z)} e^{imz} \text{ in upper half-plane}\right) \tag{77}$$

where $m > 0$.