# Modes of Oscillation

The number of modes of oscillation available to electromagnetic waves in a cavity was central to the derivation of the Rayleigh-Jeans equation. To see how the number of modes per unit volume in the wavelength range between $\lambda$ and $\lambda + \Delta\lambda$ is determined in general, let's first consider the one-dimensional case of the allowed standing waves on a string of length $L$, for example, a guitar or violin string, stretched between two points $A$ and $B$ along the $x$ axis as in Figure MO-1*a*. Standing waves can be established only for those vibrational frequencies $f$ for which the length $L$ corresponds to an integral number $n_x$ of half-wavelengths (see Figure MO-1*b*):

$$L = n_x \frac{\lambda}{2} \quad \Rightarrow \quad n_x = \frac{2L}{\lambda} \qquad \textbf{MO-1}\boldsymbol{a}$$

The question then is how many of these modes exist in the wavelength range between $\lambda$ and $\lambda + \Delta\lambda$. If $L$ is large compared with $\Delta\lambda$, then

$$\Delta n_x = -\frac{2L\,\Delta\lambda}{\lambda^2} \qquad \textbf{MO-1}\boldsymbol{b}$$

The minus sign tells us that, as $n_x$ decreases, $\Delta\lambda$ increases. In addition, the elements of the string can vibrate in any direction in the plane perpendicular to the string, so there are two degrees of freedom for each mode. Thus, we can write for $\Delta n_\ell$, the number of modes per unit volume,

$$\Delta n_L = \frac{4\Delta\lambda}{\lambda^2} \qquad \textbf{MO-2}$$

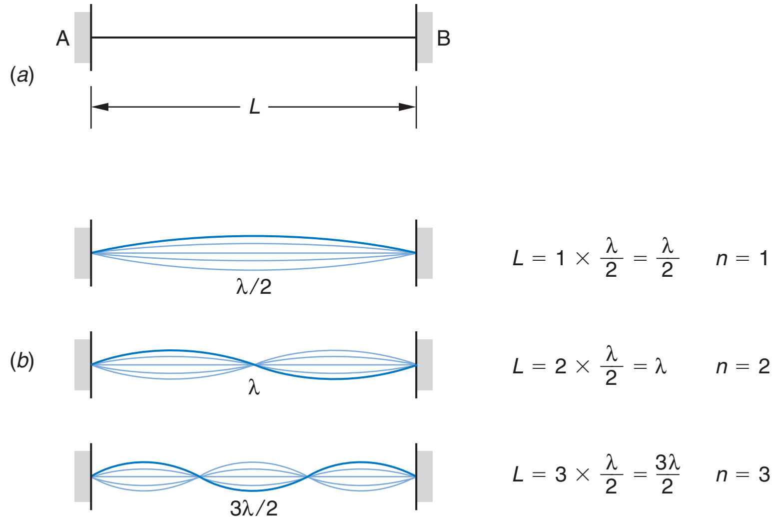
where $L$ is the "volume" in one dimension.

The description becomes more complex in two and in three dimensions. Figure MO-2 shows a square with sides of length $L$ lying in the $xy$ plane. The edges are perfect reflectors, analogous to the nodal points $A$ and $B$ for the one-dimensional string in Figure MO-1. Rays representing four electromagnetic waves with $\mathscr{E}$ perpendicular to the plane of the diagram are shown reflecting at points $\alpha$, $\beta$, $\gamma$, and $\delta$. Like the waves on the vibrating string, the four waves will form a set of standing waves only if the frequency $f$ is such that there is an integral number of half-wavelengths in both the $x$ and $y$ directions in each of the four waves. The boundary condition at the reflecting walls is $\mathscr{E} = 0$, yielding allowed solutions to the wave equation for each wave

$$\mathrm{E}(z) = C(z, L) \sin \frac{n_x \pi x}{L} \sin \frac{n_y \pi y}{L} \qquad \textbf{MO-3}$$

where $C(z, L)$ is a constant and $n_x$ and $n_y$ are integers. For example, the wave moving from $\beta$ to $\gamma$ is along the hypotenuse of a 3-4-5 right triangle. Thus, the angle $\theta = 37°$, the angle $\varphi = 53°$, and $\lambda_x$ and $\lambda_y$, the $x$ and $y$ components of $\lambda$, are given by

$$\lambda_x = \lambda / \cos\theta$$




**MO-1** (*a*) A string of length $L$ stretched between two supports $A$ and $B$. Since the string cannot move at $A$ and $B$, those two points are nodes when the string is vibrating. (*b*) The longest standing wave wavelength is $\lambda = 2L$. The next longest is $\lambda = L$, the next is $3L/2$, and so on. In general, the wavelengths of standing waves on the string are given by $L = n(\lambda/2)$, where $n$ is an integer.

$$\lambda_y = \lambda/\cos\varphi = \lambda/\sin\theta \qquad \textbf{MO-4}$$

Substituting $n_x$ and $n_y$ from the appropriate versions of Equation MO-1, we see that

$$\sin\frac{n_x \pi x}{L} = \sin\frac{2\pi x}{\lambda_x}$$

Therefore,

$$\frac{n_x \pi}{L} = \frac{2\pi}{\lambda_x} = \frac{2\pi\cos\theta}{\lambda} \qquad \textbf{MO-5}$$

Similarly,

$$\sin\frac{n_y \pi y}{L} = \sin\frac{2\pi y}{\lambda_y}$$

Therefore,

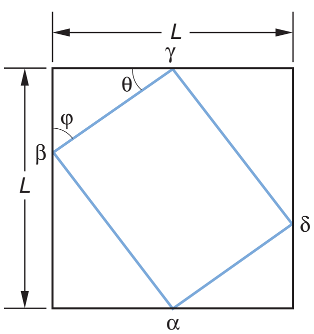
$$\frac{n_y \pi}{L} = \frac{2\pi}{\lambda_y} = \frac{2\pi\cos\varphi}{\lambda} = \frac{2\pi\cos\theta}{\lambda} \qquad \textbf{MO-6}$$

Solving Equations MO-5 and MO-6 for $n_x$ and $n_y$, respectively, squaring, and then adding the results gives

$$n_x^2 + n_y^2 = \frac{4L^2}{\lambda^2}(\cos^2\theta + \sin^2\theta) = \frac{4L^2}{\lambda^2} \qquad \textbf{MO-7}$$

which gives the allowed wavelengths.

The above method of describing the possible standing waves in a square can be extended to a cube in three dimensions. Depicting a set of standing waves in a three-dimensional cube analogous to the square in Figure MO-2 on this two-dimensional surface is not especially helpful (there are now eight waves in the set); however,




**MO-2** An example of a set of standing waves inside a two-dimensional square with sides of length $L$. The walls of the square are perfect reflectors.

note that an additional angle $\chi$ and integer $n_z$ are involved. Equation MO-8, a condition equivalent to Equations MO-5 and MO-6, also applies:

$$\frac{n_z \pi}{L} = \frac{2\pi \cos \chi}{\lambda} \qquad \textbf{MO-8}$$

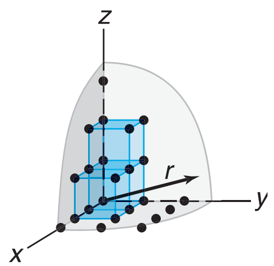
Again, solving for $n_z$, squaring, and adding the squares yields for three dimensions

$$n_x^2 + n_y^2 + n_z^2 = \frac{4L^2}{\lambda^2}(\cos^2\theta + \cos^2\varphi + \cos^2\chi)\frac{4L^2}{\lambda^2} = \frac{4L^2}{\lambda^2} \qquad \textbf{MO-9}$$

where $n_x$, $n_y$, and $n_z$ are all integers and $\lambda$ are the allowed wavelengths.

The possible combinations of $n_x$, $n_y$, and $n_z$ are at the corners of cubes in the positive octant of a sphere in $n$-space as illustrated in Figure MO-3. The number of possible combinations within a volume of radius $r$ corresponds to the number of modes $n$ for possible wavelengths $\lambda$ larger than a given value $\lambda_{\text{min}}$, where $r = 2L/\lambda_{\text{min}}$. Thus, we obtain

$$n = \left(\frac{4}{3}\pi r^3\right) \times \frac{1}{8} = \left(\frac{4}{3}\pi\frac{8L^3}{\lambda^3}\right) \times \frac{1}{8} = \frac{4\pi L^3}{3\lambda^3} \qquad \textbf{MO-10}$$




**MO-3** The $n$-space plot of $n_x$, $n_y$, and $n_z$. Each combination of possible values is a point in the space located at a corner of a cube. The number of $n$ values inside the shaded octant is that for which $n_x^2 + n_y^2 + n_z^2 < r^2$.

The number of modes with wavelengths between $\lambda$ and $\lambda + d\lambda$ is given by

$$dn = \frac{4\pi L^3}{\lambda^4}d\lambda \qquad \textbf{MO-11}$$

where we have ignored the minus sign arising from the differentiation; that is, we are considering $dn$ and $d\lambda$ both as positive (see the comment following Equation MO-1*b*). Dividing by the volume $L^3$ and noting that each allowed wavelength has two possible polarizations, we obtain the number of modes per unit volume with wavelengths between $\lambda$ and $\lambda + d\lambda$:

$$n(\lambda)d\lambda = \frac{8\pi}{\lambda^4}d\lambda \qquad \textbf{MO-12}$$

Since $c = f\lambda$ and $d\lambda = (c/f^2)df$, again ignoring the minus sign, the number of modes per unit volume with frequencies between $f$ and $f + df$ is

$$g(f) = \frac{8\pi f^2}{c^3}df \qquad \textbf{MO-13}$$