Problem 1: Algebraic manipulation of imaginary numbers
The attached appendix B is from "Principles of Physics," J.B. Marion and W.F. Hornyak, Saunders College Publishing, New York 1984
(see also Appendix D in Tipler, Ch. 3 in Hirose and Langren, class notes)

Simplify expressions a) - d) and write the answer in the Cartesian (x+iy) form:

a) $\left(\sqrt{2}-i\right)-i\left(1-i\sqrt{2}\right)$
b) $(2-3i)(-2+i)$
c) $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$
d) $(i-1)^4$

Find the value of the magnitude (ρ) and the argument (θ) for expressions f)-g)

e) 1+i
f) $\dfrac{-2}{1+i\sqrt{3}}$
g) $\left(\sqrt{3}-i\right)^6$

Use the polar form ($\rho e^{i\theta}$) to prove the equalities h) - j):

h) $\dfrac{5i}{2+i}=1+2i$
i) $(-1+i)^7=-8(1+i)$
j) $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$ {this equality is known as DeMoivre's theorem}

k) Without using a calculator, fill in the two smallest, positive values of the angles (express as multiples of π) corresponding to the values of $\tan(\theta)$ in the following table:

$\tan(\theta)$	Θ (two values, expressed as multiples of π)
0	
$\sqrt{3}/3$	
1	
$\sqrt{3}$	
$\pm\infty$	
$-\sqrt{3}$	
-1	
$-\sqrt{3}/3$	

APPENDIX B

COMPLEX NUMBERS

Basic Definitions. A *complex number* is an expression of the form $a + ib$, where a and b are ordinary real numbers and i is the imaginary unit defined by $i^2 = -1$. The *real part* of the complex number $a + ib$ is a, and the *imaginary part* is b.

Graphic Representations of Complex Numbers. A complex number may be plotted as a point in a *complex plane* defined by a horizontal (real) x-axis and a vertical (imaginary) y-axis. A general complex number may then be represented as $z = x + iy$. (In these discussions we reserve the letter z to represent a complex quantity.) Figure B–1, called an *Argand diagram,* shows several points plotted in the complex plane.

Another convenient way to represent complex numbers is in polar form, namely,

$$z = \rho(\cos\theta + i\sin\theta) \tag{B–1}$$

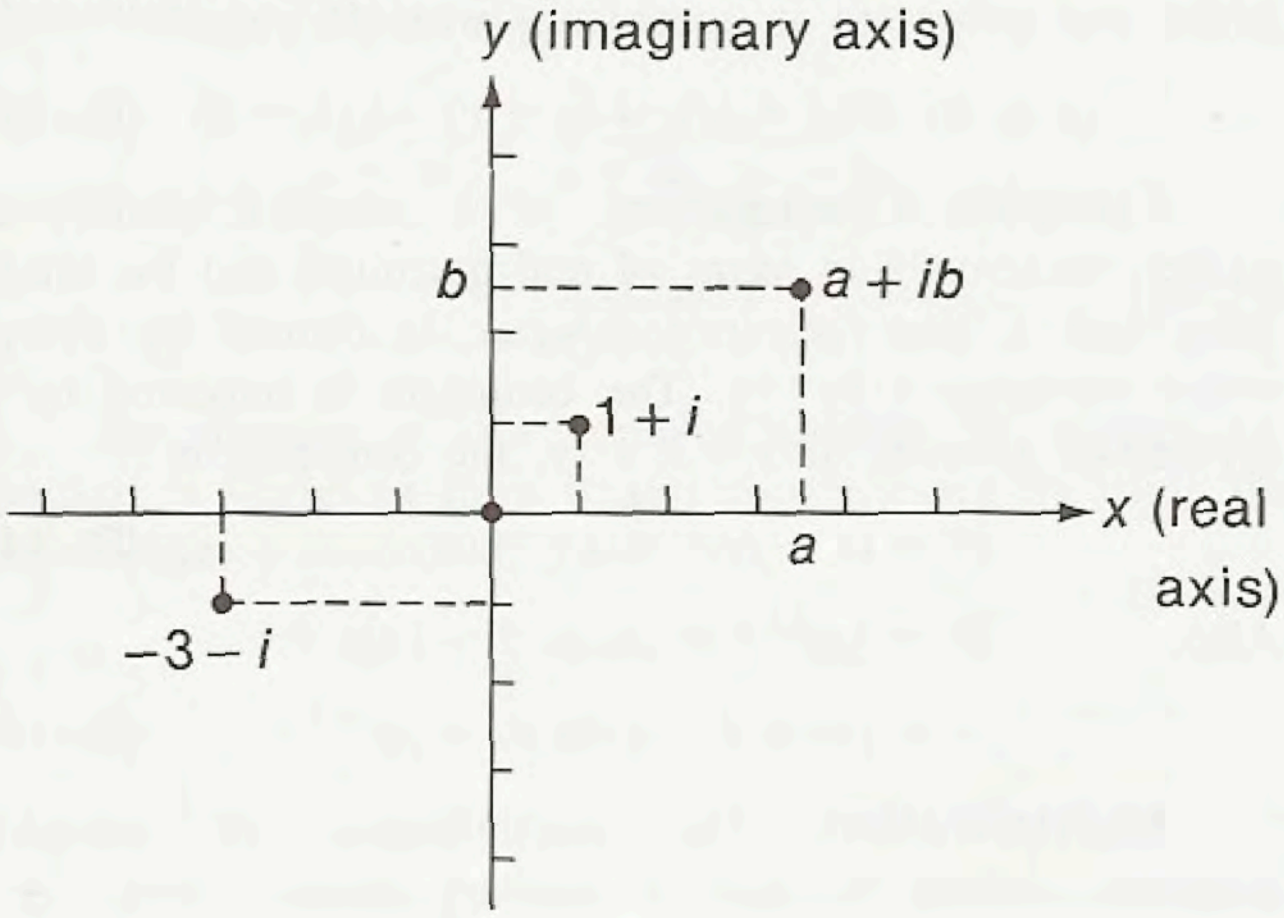
where

$$\left.\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \end{aligned}\right\} \tag{B–2}$$

Also,

$$\rho = |z| = \sqrt{x^2 + y^2} \tag{B–3}$$

Fig. B–1. An Argand diagram representation of several complex numbers $z = x + iy$.



and*

$$\tan\theta = \frac{y}{x} \tag{B–4}$$

The various polar quantities are illustrated in Fig. B–2.

The quantity ρ is the *absolute value* (or *amplitude*) of z, $\rho = |z|$. The angle θ, which is measured counterclockwise from the (real) x-axis, is defined in the domain,† $0 \leqslant 0 \leqslant 2\pi$, and is referred to as the *argument* (or *phase*) of z.

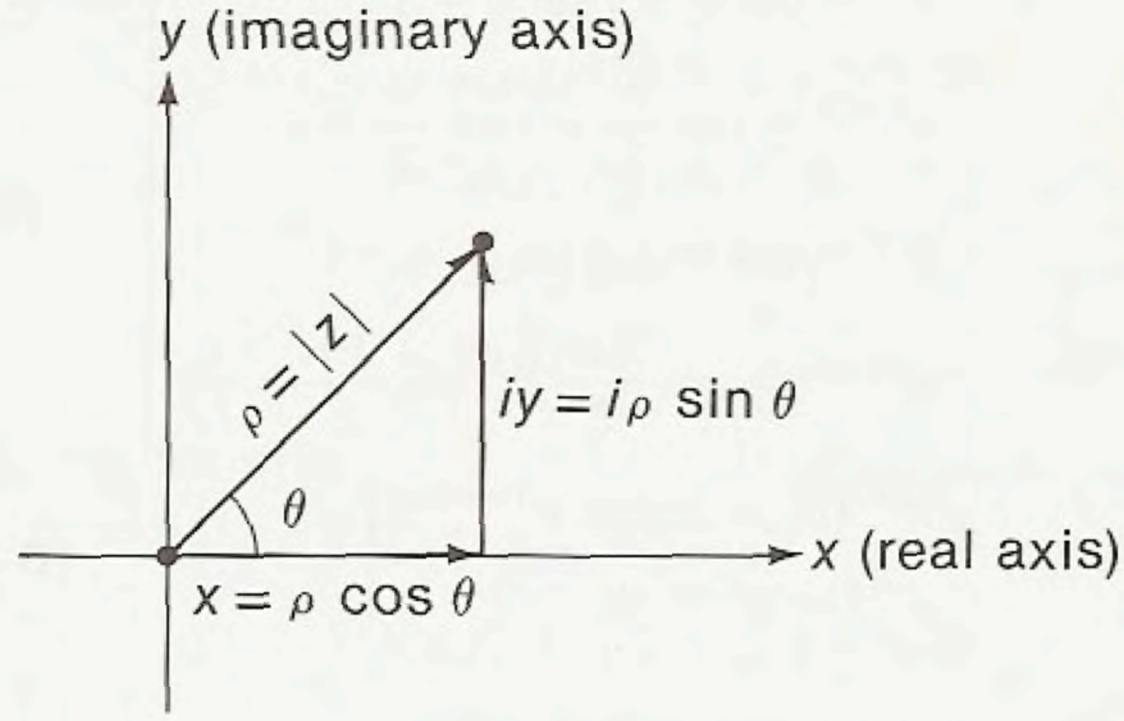
An alternative form for writing a complex number (due to Euler) is

$$z = \rho e^{i\theta} \tag{B–5}$$

This form is obtained by noting that we can *formally* express the exponential function of a complex number, $\zeta = i\theta$, in terms of the power series

$$e^{\zeta} = 1 + \zeta + \frac{\zeta^2}{2!} + \frac{\zeta^3}{3!} + \cdots + \frac{\zeta^n}{n!} + \cdots$$

Fig. B–2. The polar representation of a complex number $z = x + iy$.



*The possible sign ambiguity that results when calculating $\theta = \tan^{-1}(y/x)$ is resolved by noting the quadrant in which $x + iy$ lies. Recall the similar problem encountered in the discussion of vectors in two dimensions (Section 5–4).

†The domain of θ is sometimes defined to be $-\pi \leqslant \theta \leqslant \pi$.

$$e^{\zeta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots$$

Now, we have $i^2 = 1$, $i^3 = -i$, $i^4 = 1$, and so forth. Thus,

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

We also have

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

Hence,

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \textbf{(B–6)}$$

so that

$$z = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta) \qquad \textbf{(B–7)}$$

To be more general, we can allow ρ and θ to become complex. For example, suppose that $\theta = \alpha + i\beta$, where α and β are real numbers. Then, with $\rho = 1$, we have

$$\begin{aligned} e^{i\theta} &= e^{i(\alpha+i\beta)} = e^{i\alpha}e^{-\beta} \\ &= e^{-\beta}(\cos\alpha + i\sin\alpha) \end{aligned} \qquad \textbf{(B–8)}$$

Notice that

$$\begin{aligned} |e^{i\theta}|^2 &= |\cos\theta + i\sin\theta|^2 \\ &= \cos^2\theta + \sin^2\theta = 1 \end{aligned}$$

Thus, the quantity $e^{i\theta}$, with unit magnitude, is a convenient way to specify the phase of a complex number. Accordingly, $e^{i\theta}$ is sometimes called a *phasor* (or *rotator*) and is particularly useful when $\theta = \omega t$, for then $e^{i\theta} = e^{i\omega t}$ represents a rotation at a rate ω about the origin, as indicated in Fig. B–3.

Some useful results are

$$\left.\begin{aligned} e^{i0} &= \cos 0 + i\sin 0 = 1 \\ e^{i(\pi/2)} &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \\ e^{i\pi} &= \cos\pi + i\sin\pi = -1 \\ e^{i(3\pi/2)} &= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i \end{aligned}\right\} \qquad \textbf{(B–9)}$$

$$\left.\begin{aligned} e^{i\theta'}\ (\theta' > 2\pi) &= e^{i(\theta'-2\pi)} \\ e^{-i\theta} &= e^{i(2\pi-\theta)} \end{aligned}\right\} \qquad \textbf{(B–10)}$$

Thus,

$$e^{-i(\pi/2)} = e^{i(3\pi/2)} = -i$$

The Euler Formulas. Using Eq. B–6, namely,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we can write

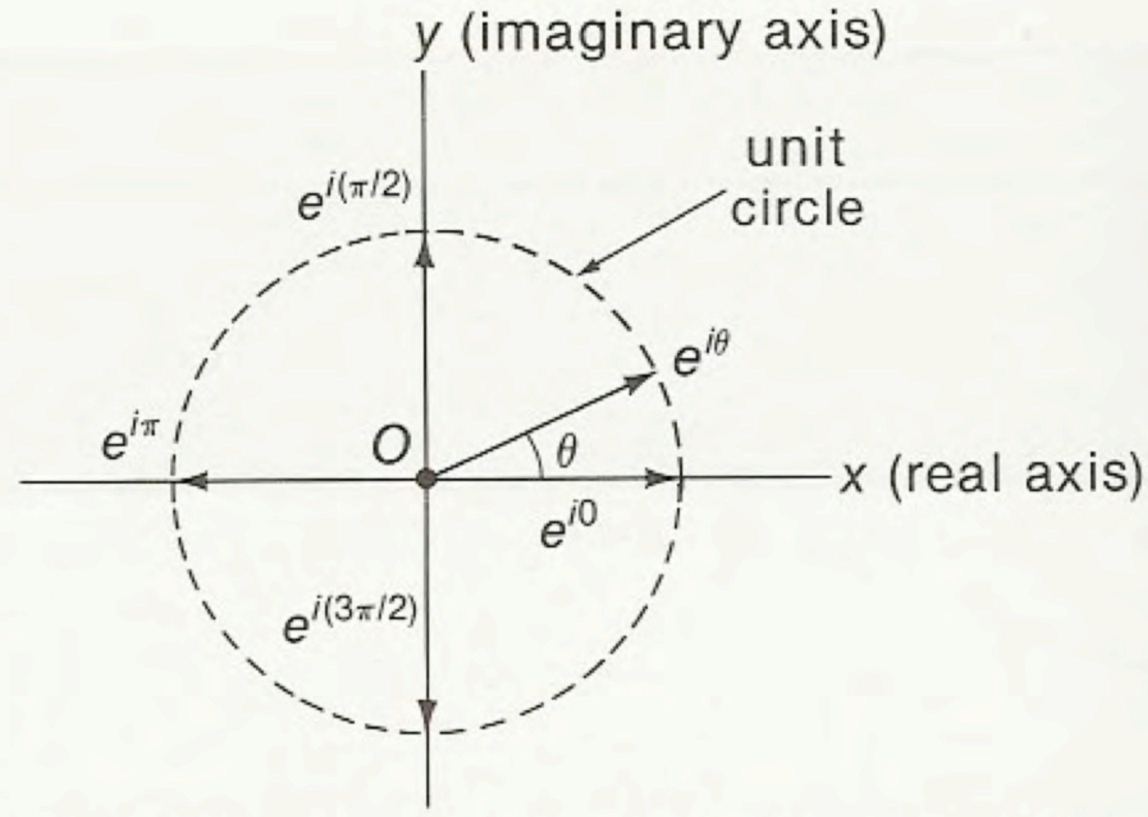


Fig. B–3. Use of the phasor $e^{i\theta}$ to represent rotation.

$$\begin{aligned} e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \end{aligned}$$

Adding these two expressions and solving for $\cos\theta$, we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \textbf{(B–11)}$$

Similarly, by subtracting we find

$$\sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i} \qquad \textbf{(B–12)}$$

Equations B–11 and B–12 are known as the *Euler formulas*.

Equality. Two complex quantities are *equal* if and only if the real parts and the imaginary parts are separately equal. Thus, if we have

$$a + ib = c + id$$

then it is required that $a = c$ and $b = d$. A complex quantity is zero if and only if both the real part and the imaginary part are separately equal to zero.

Addition and Subtraction. Complex quantities are added and subtracted in accordance with the rule

$$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d) \qquad \textbf{(B–13)}$$

Complex Conjugation. If a complex quantity is written exclusively in terms of real quantities and the imaginary unit i, the *complex conjugate* is formed by everywhere replacing i by $-i$. The conjugate is indicated by a superscript asterisk. If $z = x + iy$, the conjugate is

$$z^* = (x + iy)^* = x - iy \qquad \textbf{(B–14)}$$

Also,

$$\begin{aligned} z^* &= (\rho e^{i\theta})^* = \rho(\cos\theta + i\sin\theta)^* \\ &= \rho(\cos\theta - i\sin\theta) = \rho e^{-i\theta} \end{aligned} \qquad \textbf{(B–15)}$$

Multiplication. The multiplication of complex quantities follows the rules of ordinary algebra. Thus,

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd$$
$$= (ac - bd) + i(ad + bc) \quad \textbf{(B-16)}$$

In Euler form we have

$$\left(\rho_1 e^{i\theta_1}\right)\left(\rho_2 e^{i\theta_2}\right) = \rho_1\rho_2 e^{i(\theta_1+\theta_2)} \quad \textbf{(B-17)}$$

If a complex quantity, $z = e^{i\theta}$, is multiplied by $e^{i\phi}$ (ϕ real), the magnitude is unchanged but the phasor is rotated counterclockwise by an angle ϕ in the Argand diagram. Thus,

$$z = \rho e^{i\theta} e^{i\phi} = \rho e^{i(\theta+\phi)} \quad \textbf{(B-18)}$$

Absolute Values. The square of the absolute value of a complex quantity (that is, $|z|^2$, which is a *real* quantity) is obtained by multiplying the quantity by its complex conjugate. Thus,

$$zz^* = (a + ib)(a - ib) = a^2 - iab + iba - i^2b^2$$
$$= a^2 + b^2 = |z|^2 \quad \textbf{(B-19)}$$

In Euler form, we have

$$zz^* = (\rho e^{i\theta})(\rho e^{-i\theta}) = \rho^2 = |z|^2 \quad \textbf{(B-20)}$$

Division and Rationalization of a Complex Denominator. The division of complex quantities is most easily given in Euler form, namely,

$$\frac{z_1}{z_2} = \frac{\rho_1 e^{i\theta_1}}{\rho_2 e^{i\theta_2}} = (\rho_1/\rho_2) e^{i(\theta_1-\theta_2)} \quad \textbf{(B-21)}$$

Setting $z_1 = 1$, we have for the reciprocal of a complex quantity,

$$z^{-1} = \frac{1}{\rho e^{i\theta}} = \rho^{-1} e^{-i\theta} \quad \textbf{(B-22)}$$

(• Can you use this result to show that $i^{-1} = -i$?)

If a complex quantity is expressed in Cartesian form, the reciprocal is obtained by a process called *rationalization of the denominator*. This consists of multiplying both numerator and denominator by the complex conjugate of the denominator. Thus,

$$\frac{1}{a + ib} = \frac{1}{a + ib} \cdot \frac{a - ib}{a - ib}$$
$$= \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2} \quad \textbf{(B-23)}$$

The division of one complex quantity by another expressed in Cartesian form is also accomplished by using the rationalization procedure. Thus,

$$\frac{c + id}{a + ib} = \frac{ac + bd}{a^2 + b^2} + i\frac{ad - bc}{a^2 + b^2} \quad \textbf{(B-24)}$$

De Moivre's Theorem. Using the Euler form of a complex quantity, the nth power is expressed as

$$z^n = (\rho e^{i\theta})^n = \rho^n e^{in\theta}$$
$$= \rho^n(\cos n\theta + i \sin n\theta) \quad \textbf{(B-25)}$$

when n is a positive integer. If $\rho = 1$, we have de Moivre's theorem, namely,

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i \sin n\theta \quad \textbf{(B-26)}$$

Any multiple of 2π can be added to the angle θ without affecting the value of $e^{i\theta}$; that is,

$$e^{i\theta} = e^{i(\theta+2\pi k)}$$

where k is any integer. Then, the general expression for $z^{1/n}$, the nth root of z, is

$$z^{1/n} = \rho^{1/n} e^{i(\theta+2\pi k)/n}$$
$$= \rho^{1/n}\left(\cos\frac{\theta + 2\pi k}{n} + i \sin\frac{\theta + 2\pi k}{n}\right) \quad \textbf{(B-27)}$$

The n distinct roots of z are obtained by giving k the values 0, 1, 2, . . . , $(n - 1)$, successively. Setting $z = 1$, we see that the n roots of unity are spaced uniformly around the unit circle in an Argand diagram.

The Projection Operators, Re and Im. The operators **Re** and **Im** project the real and imaginary parts, respectively, of the complex quantity on which they act. Thus,

$$\left.\begin{aligned}\mathbf{Re}\,(a + ib) &= a\\ \mathbf{Im}\,(a + ib) &= b\end{aligned}\right\} \quad \textbf{(B-28)}$$

Notice that

$$z_1 z_2^* + z_1^* z_2 = \rho_1\rho_2(e^{i(\theta_1-\theta_2)} + e^{-i(\theta_1-\theta_2)})$$
$$= 2\rho_1\rho_2 \cos(\theta_1 - \theta_2)$$
$$= 2\mathbf{Re}(\rho_1\rho_2 e^{i(\theta_1-\theta_2)})$$
$$= 2\mathbf{Re}(z_1 z_2^*) \quad \textbf{(B-29)}$$

Then, we can write

$$|z_1 + z_2|^2 = (z_1 + z_2)(z_1^* + z_2^*)$$
$$= z_1 z_1^* + z_1 z_2^* + z_1^* z_2 + z_2 z_2^*$$
$$= |z_1|^2 + |z_2|^2 + 2\mathbf{Re}(z_1 z_2^*) \quad \textbf{(B-30)}$$