Homework #7

due Thursday March 29

1. Hirose & Lonngren Chapter 6 #4
2. A stretched string of mass m, length L, and tension T is driven by two sources (wave generators), one at each end. The sources both have the same frequency ν and amplitude A, but are exactly 180 degrees out of phase with respect to one another. (Each end is an antinode.) What is the smallest value of ω that is consistent with stationary vibrations of the string? *Hint: Draw it first.*
3. Hirose & Lonngren Chapter 6 #6 (d) Also, find the fraction of the wave energy reflected.
4. Revisit Homework 3 #6. Write a paragraph explaining your result in terms of waves.
5. *The particle in a box.* We have talked in class about how many of the laws of quantum mechanics can be derived simply from the properties of waves. One classic problem in quantum mechanics, known as "the particle in a box" or "the particle in an infinite potential well" is completely analogous to a guitar string with the ends nailed to a fret, and we can apply what we learned about standing waves to this problem to show that energy is quantized with nothing more than a bit of 3rd grade math.

   Pretend like you have a quantum mechanical particle (like an electron, for example, all that matters is that it has some wavelike properties), and the poor little particle is stuck in a one dimensional valley between two cliffs. The cliffs are a distance L apart, and infinitely high (it would take an infinite potential energy to climb them), and it therefore can't escape.

   (a) Draw the allowed wave patterns of the particle and write some quantization rules. *Hint: This is exactly the problem of a guitar string. The amplitude must be zero at the edge of the cliff, thus only certain wavelengths are allowed. What wavelengths are they?* Use n to denote an integer in your answer. For example, in Figure 6.3 in your book, one could describe the various harmonics with $L = n\lambda/2$ where $n = 1, 2, 3, ...$

   (b) Next, the momentum of the particle is $p = \hbar k$. Here, $\hbar$ is a constant, and k is our familiar wavenumber. We also know from freshman physics that the energy of a particle is given by $E = \frac{1}{2}mv^2 = p^2/2m$. What is the expression for the energy of the various energy levels?

   (c) How is the energy of the excited states of the particle related to that of the ground state? *In quantum-speak, the word "ground state" is used analogously*

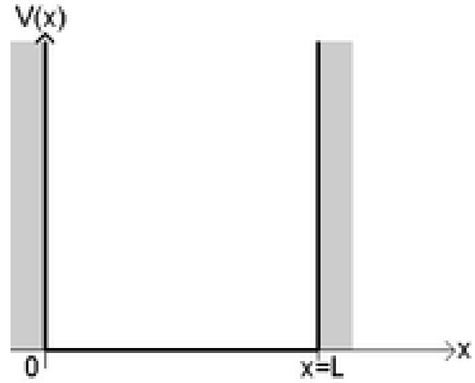

Figure 1: Illustration of an infinite potential well.

to "fundamental", and "excited state" corresponds to "harmonic". (Ans. $E_n = n^2 E_1$.)

6. *A bit more on quantum mechanics and fourier series. Localizing an electron and the uncertainty principle.* Open the spreadsheet Gaussian.xls. The sheet will add cosine functions. The values of k are fixed but the values of n and A are not. Note we are writing this in a form where $k = 2n\pi/L$. In this spreadsheets, L=6 units.

   (a) Explain why sine functions are not included on the spreadsheet.

   (b) Enter the parameters for the constant term. Why did you choose this answer?

   (c) By adding more functions to the spreadsheet, attempt to localize the electron in the given region.

   (d) Were you able to localize the electron? What would you see outside the region shown?

   (e) In the graph window, double click on the "3" that indicates the scale of the x axis. Choose the *Scale* tab. Chnage both the *Maximum* and *Minimum* values from -3 to -6. What do you observe?

   (f) How many terms would you need so that the electron is truly localized? Explain.

7. **Extra credit problem. Work it for 5 extra points. It's deceptively easy.** There is a classic problem in quantum mechanics that is very much analogous to the situation where we have a wave propagating along a string when it hits a boundary in the string density. In this case, an electron (which behaves like a wave) is incident upon a "step potential" (see Figure 2). The wavefunction of the incident electron can be described as

$$\Psi(x,t) = A_i e^{i(k_1 x - \omega t)} \tag{1}$$

and it has some kinetic energy K that is greater than the potential U of the barrier. After the electron crosses the barrier the frequency stays the same, but it's velocity is much less. That's because part of its energy was consumed in climbing the barrier. (Just like if you tried to coast up a hill on a bike after getting a running start, you would be going much slower once you reached the top.) Since we know that v, ω, and k are related, you would expect that there is a different k for the transmitted wave on the right side of the barrier:

$$\Psi(x,t) = Be^{i(k_2 x - \omega t)} \tag{2}$$

Call it k_2. Of course there could also be a reflected wave in this case, and it will be going in the opposite direction from the incident wave. When using complex exponentials, you can take care of that by reversing the sign of the wavenumber like so

$$\Psi(x,t) = A_r e^{i(-k_1 x - \omega t)} \tag{3}$$

(a) What are the boundary conditions that have to be satisfied at the edge of the step? Write them in mathematical form.

(b) Find the amplitudes of the transmitted and reflected waves in terms of the incident wave.

(c) Find the reflection coefficient (analogous to equation 6.30 in your book).

(d) Name several ways this is analogous to the string.

You might be scratching your head because you know that electrons are indivisible, and a single electron can't be transmitted and reflected. In fact, you will find in your quantum class that these coefficients are related to the probability that an electron will be reflected or transmitted.

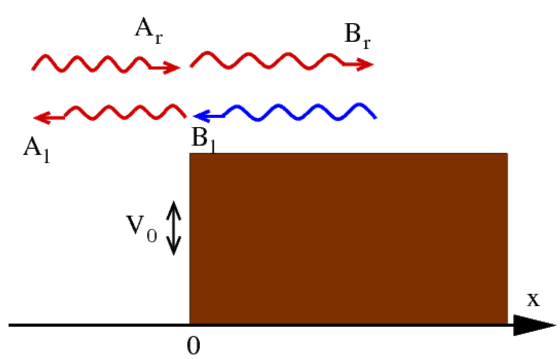

Figure 2: Illustration of a step potential.