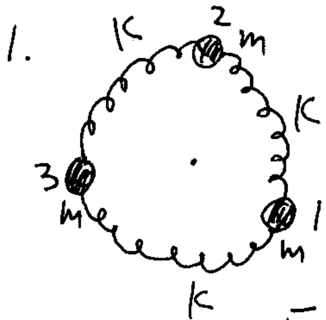
Solution to Midterm Exam

1.



x_i: displacement of mass i.

$$\mathcal{L} = T - V$$

$$= \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{k}{2}\left[(x_2 - x_1)^2 + (x_3 - x_2)^2 + (x_1 - x_3)^2\right]$$

From $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \Rightarrow$

equations of motion:

$$\ddot{x}_1 + \frac{2k}{m}x_1 - \frac{k}{m}(x_2 + x_3) = 0 \qquad *1$$

$$\ddot{x}_2 + \frac{2k}{m}x_2 - \frac{k}{m}(x_1 + x_3) = 0 \qquad *2$$

$$\ddot{x}_3 + \frac{2k}{m}x_3 - \frac{k}{m}(x_1 + x_2) = 0 \qquad *3$$

make ansatz: $x_i = A_i e^{i\omega t}$. $\quad i = 1, 2, 3$.

Plug in $*1 \longrightarrow *3$. $\quad \dot{x}_i = i\omega A_i e^{i\omega t} \quad \ddot{x}_i = -\omega^2 A_i e^{i\omega t}$

$$(-\omega^2 \cancel{A_m} + \frac{2k}{m})A_1 - \frac{k}{m}(A_2 + A_3) = 0$$

$$-\frac{k}{m}A_1 + (-\omega^2 + \frac{2k}{m})A_2 - \frac{k}{m}A_3 = 0$$

$$-\frac{k}{m}A_1 - \frac{k}{m}A_2 + (-\omega^2 + \frac{2k}{m})A_3 = 0$$

if these eqns with non-trivial solution, the determinant of the coefficients of A_1, A_2, A_3 Vanishes.

$$\therefore \begin{vmatrix} -\lambda + 2 & -1 & -1 \\ -1 & -\lambda + 2 & -1 \\ -1 & -1 & -\lambda + 2 \end{vmatrix} = 0 \qquad \text{where } \lambda = \frac{\omega^2}{\omega_0^2} \quad \omega_0^2 = \frac{k}{m}$$

$\Rightarrow \lambda_1 = \omega = 0 \quad$ a rotation of $m_1, m_2, m_3 \Rightarrow A_1 = A_2 = A_3$