4. Let η_i be the displacement of block i from equilibrium, The Potential energy of the system is.

$$V = \frac{1}{2}k(\eta_1-\eta_2)^2 + \frac{1}{2}k(\eta_2-\eta_3)^2$$

$$T = \frac{1}{2}m(\dot\eta_1^2+\dot\eta_2^2+\dot\eta_3^2)$$

$\therefore$ Lagrangian $L = T-V$.

$$\therefore L = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(T_{ij}\dot\eta_i\dot\eta_j - V_{ij}\eta_i\eta_j)$$

where $\overleftrightarrow{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}$ $\quad \overleftrightarrow{V} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$

Using Lagrange's equation, we find the equation of motion:

$$\overleftrightarrow{T}\ddot{\vec\eta} + \overleftrightarrow{V}\vec\eta = 0. \qquad \text{here } \vec\eta = (\eta_1, \eta_2, \eta_3)^T.$$

let $\vec\eta(t) = \vec a_j e^{i\omega_j t}$ where a_j is time-independent.

$$\Rightarrow (\overleftrightarrow{V} - \omega_j^2\overleftrightarrow{T})\vec a_j = 0 \qquad *$$

In order for a nontrivial solution to exist, we must have

$$\det|\overleftrightarrow{V} - \omega_j^2\overleftrightarrow{T}| = 0.$$

This leads to a cubic equation of $\omega^2 \Rightarrow \omega_1^2 = 0$, $\omega_2^2 = k/m$. and $\omega_3^2 = 3k/m$. Substituting ~~these~~ frequencies into equation $*$, allows us to solve for the three normal modes, for which we choose the normalisation prescription

$$\vec a_i^T\overleftrightarrow{T}\vec a_i = 1.$$

or choose $\vec a_i^T\overleftrightarrow{T}\vec a_j = \delta_{ij}$.