3\. a) $x(t) = (A+Bt)e^{-\gamma t/2}$ with $\gamma = 2\omega_0$

$$\dot{x} = Be^{-\gamma t/2} - \gamma/2\,(A+Bt)e^{-\gamma t/2}$$

$$\ddot{x} = -\frac{\gamma B}{2}e^{-\gamma t/2} - \gamma/2\,Be^{-\gamma t/2} + \frac{\gamma^2}{4}(A+Bt)e^{-\gamma t/2}$$

$$= -\gamma Be^{-\gamma t/2} + \frac{\gamma^2}{4}(A+Bt)e^{-\gamma t/2}$$

$$m\ddot{x} + \gamma\dot{x} + \omega_0^2 x = e^{-\gamma t/2}\left[\cancel{-\gamma B} + \frac{\gamma^2}{4}(A+Bt) + \cancel{\gamma B} - \frac{\gamma^2}{2}(A+Bt) + \omega_0^2(A+Bt)\right]$$

$$= e^{-\gamma t/2}(A+Bt)\left(\underbrace{-\frac{\gamma^2}{4} + \omega_0^2}_{\to 0}\right) = 0$$

for critically damped.

b) $x_0 = x(t=0) = A$

$$v_0 = B - \gamma/2\, x_0 \Rightarrow B = v_0 + \frac{\gamma}{2}x_0 \xrightarrow{B=0} v_0 = -\frac{\gamma}{2}x_0$$

c) From a), b)

$$x(t) = (x_0 + (v_0 + \frac{\gamma}{2}x_0)t)e^{-\gamma t/2}$$

~~[illegible]~~.

Let's find the minimum of $x(t)$

$$\dot{x}(t) = -\frac{\gamma}{2}(x_0 + (v_0 + \frac{\gamma}{2}x_0)t)e^{-\gamma t/2} + (v_0 + \frac{\gamma}{2}x_0)e^{-\gamma t/2} = 0$$

$$t_{min} = \frac{2v_0/\gamma}{v_0 + \frac{\gamma}{2}x_0} \Rightarrow t \text{ for } x \text{ minimum}$$

$x_{min} = (x_0 + \frac{2v_0}{\gamma})e^{-\gamma t/2}$ and we want this expression to be negative

$$\Rightarrow +x_0 + \frac{2v_0}{\gamma} < 0$$

$$\boxed{v_0 < -\frac{\gamma x_0}{2}} \Rightarrow \boxed{v_0 < -\omega_0 x_0}$$

This is equivalente to $B<0$ which is necessary for $x(t_{min})$ to be actually a minimum and not a maximum.