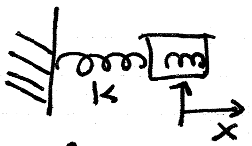
NON-LINEAR SYSTEMS

Let us go back to our first oscillator, the simple harmonic oscillator. Its equation of motion is:

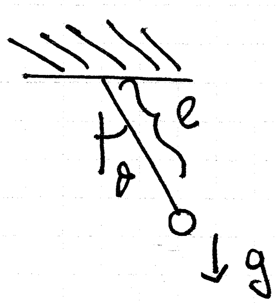
$$\ddot{x} + \omega^2 x = 0$$

Examples:



$\omega^2 = k/m$

$$\ddot{x} + \frac{k}{m} x = 0$$



$\omega^2 = g/\ell$

small ϑ

$$\ddot{\vartheta} + \frac{g}{\ell}\vartheta = 0$$

The solution of the undampened, not-forced, oscillator is:

$$x(t) = A\cos(\omega_0 t + \delta)$$

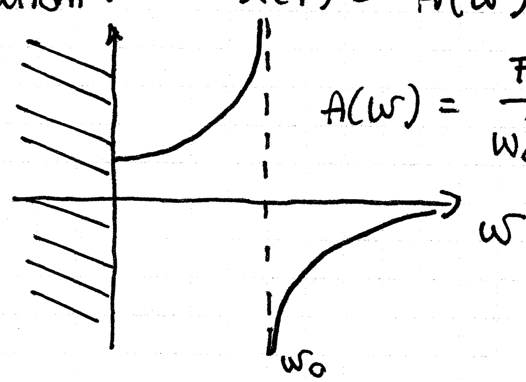
where A and δ can be fixed by the initial conditions $x_0 \equiv x(t=0)$ and $v_0 \equiv \dot{x}(t=0)$.

In the case of a forced oscillator, without damping, we have:

assume : $F(t) = F_0 \cos(\omega t)$

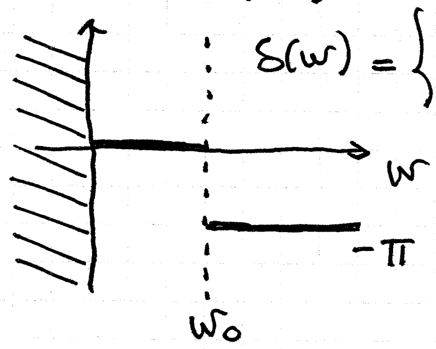
equation : $\ddot{x} + \omega_0^2 x = F(t)$

solution : $x(t) = A(\omega) \cdot \cos(\omega t + \delta(\omega))$



$$A(\omega) = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

$$\delta(\omega) = \begin{cases} 0, & \omega < \omega_0 \\ -\pi, & \omega > \omega_0 \end{cases}$$



It is interesting to note that $x(t)$ oscillates with frequency ω, regardless of the natural frequency ω_0. The relation between the natural frequency ω_0 and the driving frequency ω dictates the amplitude and phase of $x(t)$ with respect to the driving force $F(t)$.

Now, let us assume that the system is non-linear, i.e., that the equation of motion is:

$$\ddot{x} + s(x) = F(t)$$

$$s(x) = \omega_0^2 x + a x^2 + b x^3 + \ldots$$

It is not a strange assumption. Remember the pendulum: the actual equation of motion is:

$$\ddot{\vartheta} + \frac{g}{l} \sin\vartheta = 0$$

$$\sin\vartheta \approx \vartheta - \vartheta^3/3! + \vartheta^5/5!$$

we kept only the first term, ϑ, assuming that angles are small, but we could improve our approximation by keeping also the second term in the expansion, thus obtaining the non-linear equation:

$$\ddot{\vartheta} + \frac{g}{l}\vartheta - \frac{g}{6l}\vartheta^3 = 0$$

How can we solve it? Well, let us assume again a solution in the form:

$$x(t) = A\cos(\omega t + \delta)$$

and let us see if it can solve:

$$\ddot{x} + \omega_0^2 x + s x^3 = \frac{F_0}{m}\cos(\omega_0 t)$$

Let us assume a solution in the form:

$$x(t) = x_0(t) + s\, x_1(t) + s^2 x_2(t) + \dots$$

and solve separately the various orders in s:

$$s^0 \; : \quad \ddot{x}_0 + \omega_0^2 x_0 = F(t)/m$$

$$s^1 \; : \quad \ddot{x}_1 + \omega_0^2 x_1 + x_0^3 = 0$$

$$s^2 \; : \quad \ddot{x}_2 + \omega_0^2 x_2 + 3x_0^2 x_1 = 0$$

The solution of the first equation is easy:

$$x_0(t) = \frac{\pm F_0}{m(\omega_0^2 - \omega^2)} \cdot \cos(\omega t) \qquad \pm : \omega < \omega_0 \text{ or } \omega > \omega_0$$

For the sake of brevity, I will only solve the second equation, and leave for (a difficult) exercise the higher orders.

$$\ddot{x}_1 + \omega_0^2 x_1 + x_0^3 = 0$$

$$x_0^3 = A^3(\omega) \cdot \cos^3(\omega t) = \frac{3A^3(\omega)}{4} \cos(\omega t) + \frac{A^3(\omega)}{4} \cos(3\omega t)$$

trigonometry!

If you want, this is a linear system, with two forces operating on it:

$$\frac{F_1(t)}{m} = -\frac{3A^3(\omega)}{4} \cos(\omega t) \quad \text{and} \quad \frac{F_2(t)}{m} = -\frac{A^3(\omega)}{4} \cos(3\omega t)$$

Therefore:

$$x_1(t) = \mp \frac{3A^3(\omega)}{4} \cdot \frac{1}{\omega_0^2 - \omega^2} \cos(\omega t) \mp \frac{A^3(\omega)}{4} \cdot \frac{1}{\omega_0^2 - 9\omega^2} \cdot \cos(3\omega t)$$

$\pm : \omega < \omega_0$ or $\omega > \omega_0$ $\qquad$ $\pm : \omega < \frac{\omega_0}{3}$ or $\omega > \frac{\omega_0}{3}$!

The final solution will be:

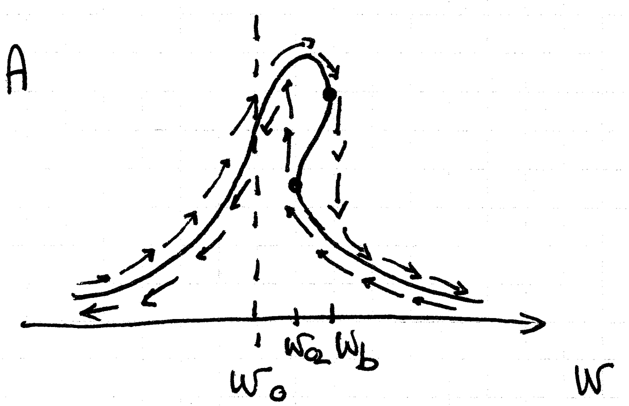
$$x(t) = \frac{\mp F_0/m}{w_0^2 - w^2}\cos(wt) + \frac{3(F_0/m)^3}{4(w_0^2-w^2)^4}\cdot s\cos(wt) \mp \frac{(F_0/m)^3}{4(w_0^2-w^2)^3}\cdot\frac{s}{(w_0^2-9w^2)}\cos(3wt)$$

First $\mp$:
$-$: $w < w_0$
$+$: $w > w_0$

Second $\mp$:
$-$: $w < w_0/3$
$+$: $w > w_0/3$ and $w < w_0$
$-$: $w > w_0$

Note that a third-order harmonic ($\cos(3wt)$) appeared. When the system is non-linear, higher-order harmonics of the natural or driving frequency can appear.

The amplitude of the $\cos(wt)$ part of the solution has an interesting behavior if $s \neq 0$. Let us take $s > 0$ ("hard" spring: the force is higher than a normal spring – the case $s < 0$ is referred to as a "soft" spring). We obtain:



If s is large enough, there are three solutions for A at a given w. This causes shock jumps and hysteresis: the arrow $\rightarrow$ indicates the values of A taken when we increase w from $w < w_0$ to $w > w_0$, while the arrow $\leftarrow$ indicates the values of A taken when we move from $w > w_0$ to $w < w_0$. Note the two jumps at w_a and w_b.