POLARIZATION

Polarization is a general feature of <u>transverse</u> waves in (at least) three dimensions. For example, the general EM wave has two polarization states, corresponding to the two directions that the electric field can point, on the plane transverse to the direction of the wave's motion. Let us take a plane EM wave moving in the $\hat{z}$ direction. I can write:

in vacuum

$$\vec{E} = E_0 \cdot \hat{n} \cdot e^{i(kz-\omega t)}$$

E_0: a number!

$\hat{n}$: a vector lying on the plane $\perp$ to the $\hat{z}$ axis

$e^{i(kz-\omega t)}$: the phase of a wave travelling along the $\hat{z}$ axis.

The wave equation imposes the condition:

$$c = v_p = v_g = \frac{\omega}{k}$$

Maxwell's equations tell us that:

$$\vec{B} = \frac{E_0}{c} \frac{\vec{k}}{|k|} \times \hat{n}\, e^{i(kz-\omega t)} \qquad \text{where } \vec{k} = k\hat{z}$$

$\hat{n}$ is a two-component ($\hat{x}$ and $\hat{y}$) unit vector (its $\hat{z}$ component is ϕ, by construction). Technically, you'd expect it to have one degree of freedom (a rotation angle around $\hat{z}$), but E_x and E_y do not necessarily have to be in phase. Therefore, $\hat{n}$ has two degrees of freedom. I can write:

$$\hat{n} = \begin{pmatrix} \cos\vartheta \\ \sin\vartheta\, e^{i\varphi} \end{pmatrix}$$

the $\hat{z}$ component has to be ϕ: the wave propagates in the $\hat{z}$ direction

Let us now check some simple cases, which will help understanding why I have two degrees of freedom, ϑ and φ.

Let us start with a simple plane wave with only an $\hat{x}$ component for the $\vec{E}$ field; i.e., $\vec{E}_o = (E_{ox}, \emptyset, \emptyset)$.

Then, I can write:

$$\vec{E} = E_o \cdot (1, 0) \cdot e^{i(kz-\omega t)}$$

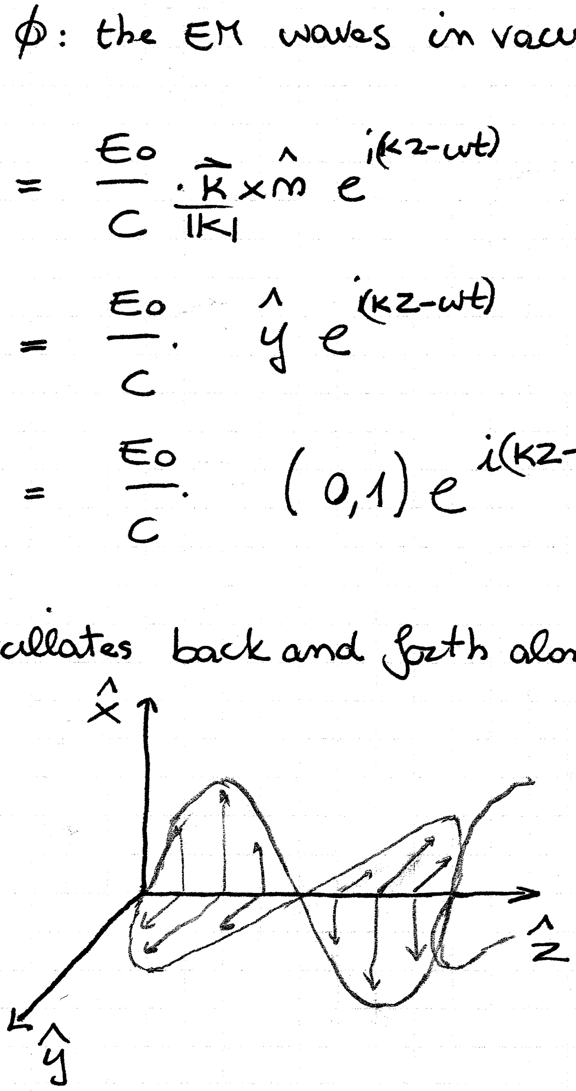
note that $\hat{n}$ <u>must</u> be a 3D vector, otherwise I cannot define $\vec{k} \times \hat{n}$. However, I keep writing it as a 2D vector because its $\hat{z}$ component is $\emptyset$: the EM waves in vacuum are transverse.

$$\vec{B} = \frac{E_o}{c} \cdot \frac{\vec{k} \times \hat{n}}{|k|} e^{i(kz-\omega t)} \qquad \vec{k} \times \hat{n} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & k \\ 1 & 0 & 0 \end{vmatrix} = k\hat{y}$$

$$= \frac{E_o}{c} \cdot \hat{y}\, e^{i(kz-\omega t)}$$

$$= \frac{E_o}{c} \cdot (0,1)\, e^{i(kz-\omega t)}$$

$\vec{E}$ oscillates back and forth along the $\hat{x}$ axis, and $\vec{B}$ along the $\hat{y}$ axis.



This is what we call

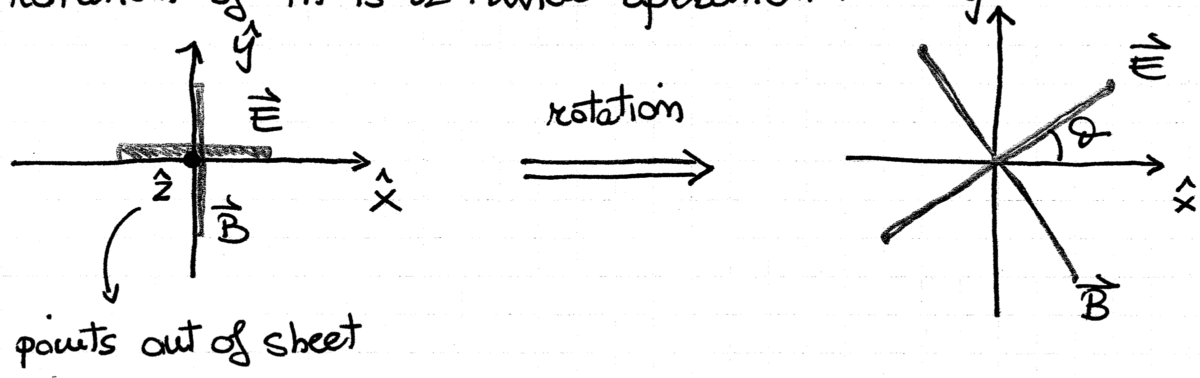
LINEAR POLARIZATION

The next most complicate polarization is actually rather trivial: we can rotate $\hat{n}$ by an angle ϑ:

$$\hat{n} = (\cos\vartheta, \sin\vartheta, \emptyset)$$

$$\underbrace{\hat{z}}_{\text{the wave direction}} \times \hat{n} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ \cos\vartheta & \sin\vartheta & 0 \end{vmatrix} = (-\sin\vartheta, \cos\vartheta, \emptyset)$$

If you look straight down from the $\hat{z}$ axis, you will see that the rotation of $\hat{n}$ is a trivial operation :



$\vec{E}$ and $\vec{B}$ are in phase : this stems from Maxwell's equations. With this polarization, also the $\hat{x}$ and $\hat{y}$ components of $\vec{E}$ and $\vec{B}$ are in phase : when E_x is max, also E_y is max.
This is <u>not</u> a requirement. In fact, if you look at Maxwell's equation, they require $\vec{B}$ and $\vec{E}$ to be orthogonal, nothing more.
Let us show that I can add a phase difference between the $\hat{x}$ and $\hat{y}$ component of $\vec{E}$, and still solve Maxwell's equations and satisfy all constraints.

$$\hat{n} = \frac{1}{\sqrt{2}}\left(1, e^{i\varphi}, 0\right)$$

: it is a unit vector, and it introduces a phase difference between E_x and E_y

$$\hat{z} \times \hat{n} = \frac{1}{\sqrt{2}}\left(-e^{i\varphi}, 1, 0\right)$$

(note that I can absorb a global phase change in the term $\exp(i(kz-\omega t))$, which is common to E_x and E_y)

$$\vec{E} = E_0 \hat{n} e^{i(kz-\omega t)}$$ becomes :

$$E_x = E_0/\sqrt{2}\; e^{i(kz-\omega t)}$$
$$E_y = E_0/\sqrt{2}\; e^{i(kz-\omega t+\varphi)}$$
$$E_z = 0$$

$$B_x = \frac{-E_0}{c}\frac{1}{\sqrt{2}} e^{i(kz-\omega t+\varphi)}$$
$$B_y = \frac{E_0}{c}\frac{1}{\sqrt{2}} e^{i(kz-\omega t)}$$
$$B_z = 0$$

The most interesting case is when E_x and E_y are 90° out of phase:

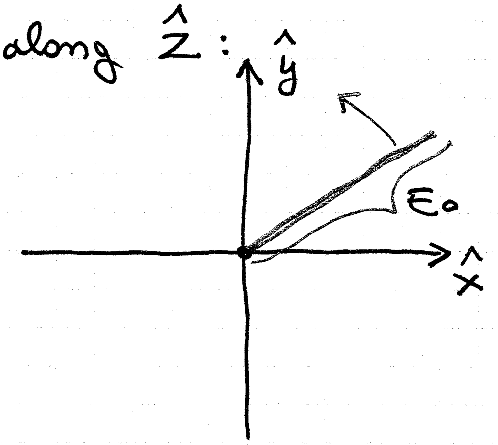
$$\hat{m} = (1, e^{\frac{\pi}{2}i}, \phi) = (1, i, 0)$$

what is going on? Let us take the real fields (and let me drop the $1/\sqrt{2}$ factor...):

$$E_x = \mathrm{Re}\, E_0\, e^{i(kz-\omega t)} = E_0 \cos(kz-\omega t)$$

$$E_y = \mathrm{Re}\, E_0\, e^{i(kz-\omega t+\pi/2)} = E_0 \cos(kz-\omega t+\pi/2) = E_0 \sin(kz-\omega t)$$

It is a 2D vector that rotates in the $\hat{x}$-$\hat{y}$ plane when the wave travels along $\hat{z}$:



if I sit at $z=0$, as the wave progresses I see the $\vec{E}$ field rotate around the $\hat{z}$ axis. If I stop time and start walking along $\hat{z}$, I see the $\vec{E}$ field draw a helix. Starting from $\hat{m} = (1, i, 0)$, you can find that $\vec{B}$ does a similar movement

Let us take $\hat{m} = (1, -i, 0)$:

$$E_x = \mathrm{Re}(E_0 e^{i(kz-\omega t)}) = E_0 \cos(kz-\omega t)$$

$$E_y = \mathrm{Re}(E_0 e^{i(kz-\omega t-\pi/2)}) = -E_0 \sin(kz-\omega t)$$

The field now rotates clockwise around $\hat{z}$. These are called CIRCULAR POLARIZATIONS:

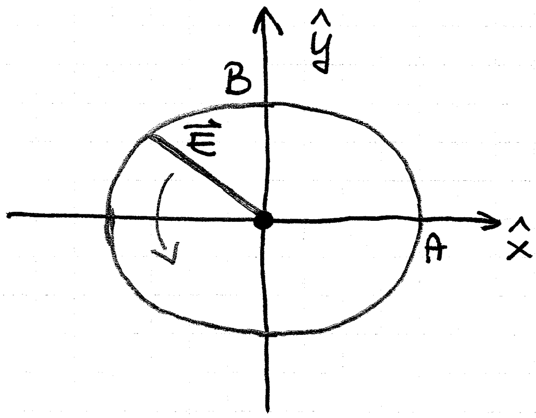
$\hat{m}_L = (1, i, 0)$: left-circular : fields go counter-clockwise

$\hat{m}_R = (1, -i, 0)$: right-circular : fields go clockwise

Finally, what if I take:

$$\hat{m} = (A, Be^{i\pi/2}, 0)$$

? This is an ELLIPTICAL POLARIZATION (note that I care only about the A/B ratio, not their absolute value: I can extract it and absorb it in the normalization of the $\hat{m}$ vector).



Finally note that I can express any generic polarization using a linear combination of either linear or circular polarization i.e., $m_x = (1,0,0)$ and $m_y = (0,1,0)$ are a base for all polarizatio[ns] and so are $m_L = (1,i,0)$ and $m_R = (1,-i,0)$.