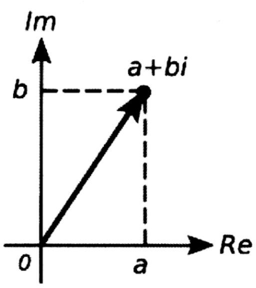
Visualizing complex numbers

by K.D. Hoffman

Student's Name:

Complex numbers can be visualized as vectors in a plane. The projection on the horizontal axis corresponds to the "real" portion of the number, and vertical (y) axis is replaced by an "imaginary" axis. It is sometimes convenient to represent such numbers in polar form. Write an expression for the number a+bi shown below in polar coordinates.

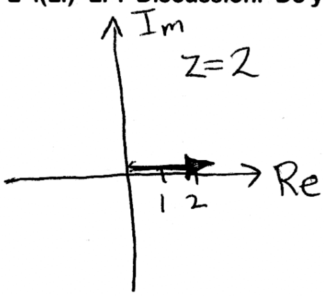


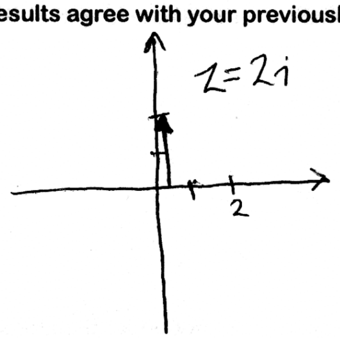
$x = r\cos\theta$ where $r = \sqrt{a^2+b^2}$

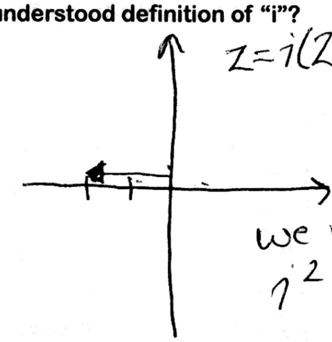
$y = r\sin\theta$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

We can expand on this polar representation by noting that multiplying a number by "i" is equivalent to a 90 degree rotation in the complex plane. In the space provided below, draw three graphs in the complex plane: z=2, z=2i, and z=i(2i)=2i². Discussion: Do your results agree with your previously understood definition of "i"?





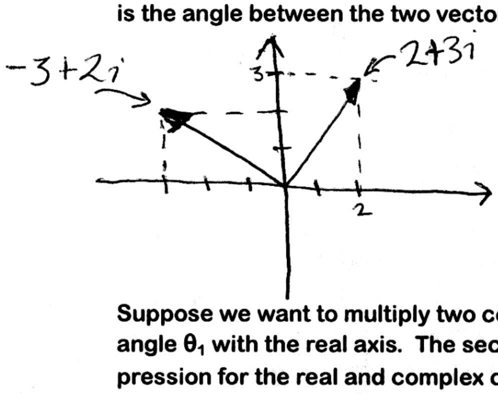


$z = i(2i) = 2i^2 = -2$ ($i^2 = -1$)

each a 90° rotation

we previously knew $i^2 = -1$

Plot the number z=2+3i as a vector in the complex plane. Next, multiply 2+3i by i and plot it on the same graph. What is the angle between the two vectors?



$i(2+3i)$

$= 2i - 3$

angle between them is 90°

Suppose we want to multiply two complex numbers. The first, z_1, is represented as vector of length r_1 and makes an angle θ_1 with the real axis. The second, z_2, has a length r_2 and makes an angle θ_2 with the real axis. Write an expression for the real and complex components of these numbers, then multiply them to find an expression for the product z_1z_2. Discussion: Qualitatively describe your result. Hint: The relations found in appendix B of your book will be useful.

$$z_1 = r_1 \cos\theta_1 + i r_1 \sin\theta_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2 \cos\theta_2 + i r_2 \sin\theta_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2 (\overbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}^{\cos(\theta_1+\theta_2)} + \underbrace{i\sin\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1}_{+ i\sin(\theta_1+\theta_2)})$$

$$= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

same form except lengths are multiplied and angles are added

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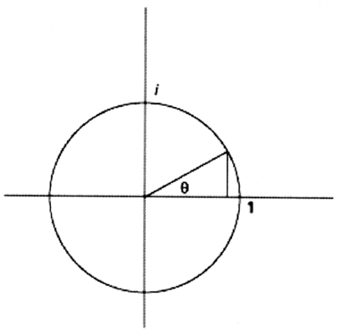
It is useful to define a unit vector in the the complex plane in polar coordinates, defined by $z=\cos\theta+i\sin\theta$. Since $|z|=1$, then using the rule you derived in the previous question, mutiplying two unit vectors is as easy as adding their angles! Compare this result to the multiplication rule for exponents, i.e., $a^{\theta 1}a^{\theta 2}=?$

$a^{\theta 1}a^{\theta 2}=a^{(\theta_1+\theta_2)}$



This leads to the nice result $e^{i\theta}=\cos\theta+i\sin\theta$, known as Euler's formula which we will prove in class on Monday. Complex number can therefore be written in the form $z=re^{i\theta}$, where r is the length of the vector, and θ is the displacement from the real axis.

Write the following in the form $z=re^{i\theta}$. a) $z = 1 + 2i$ b) $z = -3 + 2i$ c) $z = 5 - 3i$

a) $r=\sqrt{1^2+2^2}=\sqrt{5}$
$\theta=\tan^{-1}(2)=63°$
$=0.35\pi$

b) $r=\sqrt{(-3)^2+(2)^2}=\sqrt{13}$
$\theta=\tan^{-1}(-\frac{2}{3})$
$180-33=146.3°$
$=0.812\pi$

c) $r=\sqrt{(5)^2+(3)^2}=\sqrt{34}$
$\theta=\tan^{-1}(\frac{-3}{5})$
$=-30°=.172\pi$

Does the expression $x=Ae^{i\omega t}$ solve the differential equation of motion for the mass on a spring? (Try it.)

$\frac{dx}{dt}=i\omega Ae^{i\omega t}$

$\frac{d^2x}{dt^2}=(i\omega)^2Ae^{i\omega t}$
$=-\omega^2Ae^{i\omega t}$

$m\frac{d^2x}{dt^2}+kx=0$

$-m\omega^2Ae^{i\omega t}+kAe^{i\omega t}=0$

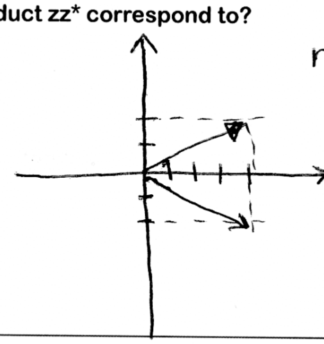
true if $\omega=\sqrt{\frac{k}{m}}$

Consider the complex number $z = 4 + 2i$. The complex conjugate of a number is found by replacing i with -i. Draw both z and z* (the complex conjugate) in the complex plane. What does the product zz* correspond to?

$z^*=4-2i$

$zz^*=(4+2i)(4-2i)$
$=16+8i-8i+4=20=|z|^2$

not that
$|z|=\sqrt{(4)^2+(2)^2}=20$



not $z=re^{i\theta}$
$z^*=re^{-i\theta}$
$zz^*=r^2$!!

Find the product z_1z_2 where $z_1 = 2 + 3i$ and $z_2 = 6 - 2i$.

$(2+3i)(6-2i)=12+18i-4i+6$
$=18+14i$