Progressive Waves

Name:______________________________

Consider three waves: y_1=A sin(5x-10t), y_2=A sin(4x-8t), and y_3=A sin(4x-9t). Do these waves satisfy the wave equation? What is the speed of each wave?

$k_1\ \omega_1 \quad k_2\ \omega_2 \quad k_3\ \omega_3$

$V_1 = \frac{\omega_1}{k_1} = \frac{10s^{-1}}{5m^{-1}} = 2m/s \quad V_2 = \frac{8s^{-1}}{4m^{-1}} = 2m/s \quad V_3 = \frac{\omega_3}{k_3} = \frac{9s^{-1}}{4m^{-1}} = 2.25m/s$

$\frac{\partial^2 y_1}{\partial x^2} = -25A\sin(5x-10t) \quad \frac{\partial^2 y_2}{\partial x^2} = -16A\sin(4x-8t) \quad \frac{\partial^2 y_3}{\partial x^2} = -16A\sin(4x-9t)$

$\frac{\partial^2 y_1}{\partial t^2} = -100A\sin(5x-10t) \quad \frac{\partial^2 y_2}{\partial t^2} = -64A\sin(4x-8t) \quad \frac{\partial^2 y_3}{\partial t^2} = -81A\sin(4x-9t)$

wave equation: $\frac{\partial^2 f}{\partial t^2} = C_w^2 \frac{\partial^2 f}{\partial x^2}$ where C_w is the wave velocity

the equation is satisfied for y_1 and y_2 if $C_w = 2$

it is satisfied for y_3 if $C_w = 2.25$

Does the superposition y_1+y_2 satisfy the wave equation? What about the superposition y_1 $+y_3$? If the answer is no for either, explain why. *(Hint: It is easier to take derivatives of the summed wave if you leave in written in the form of a sum instead of using trigonometric relations to rewrite it as a product.)*

$y_1 + y_2 = A\sin(5x-10t) + A\sin(4x-8t)$

$\frac{\partial^2}{\partial x^2}(y_1+y_2) = -25A\sin(5x-10t) - 16A\sin(4x-8t)$

$\frac{\partial^2}{\partial t^2}(y_1+y_2) = -100A\sin(5x-10t) - 64A\sin(4x-8t)$

plugging this into $\frac{\partial^2 f}{\partial t^2} = C_w^2 \frac{\partial^2 f}{\partial x^2}$

$-100A\sin(5x-10t) - 64A\sin(4x-8t) = C_w^2\{-25A\sin(5x-10t) - 16A(4x-8t)\}$

this is satisfied if $C_w^2 = 4 \Rightarrow C_w = 2$

$y_1 + y_3 = A\sin(5x-10t) + A\sin(4x-9t)$

$\frac{\partial^2}{\partial x^2}(y_1+y_3) = -25A\sin(5x-10t) - 16A\sin(4x-9t)$

$\frac{\partial^2}{\partial t^2}(y_1+y_3) = -100A\sin(5x-10t) - 81A\sin(4x-9t)$

plugging this into the wave equation:

$-100A\sin(5x-10t) - \cancel{16}A\sin(4x-9t)$

$= C_w^2\{-25A\sin(5x-10t) - 81A\sin(4x-9t)\}$

this cannot be satisfied for any C_w

this is because the two component waves are propogating at different speeds

Use $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

For these combined waves (y_1+y_2 and y_1+y_3), what is the group velocity and what is the phase velocity? ***(Recall that the group velocity is the speed of the "envelope", and the phase velocity is the speed of the wave inside the envelope. For this you will need to write the combined wave as a product.)***

$$y_1+y_2 = A\sin(5x-10t) + A\sin(4x-8t)$$

$$= 2A\,\underbrace{\sin\left(\tfrac{9}{2}x - 9t\right)}_{\text{high frequency wave}}\,\underbrace{\cos\left(\tfrac{1}{2}x - t\right)}_{\text{low frequency envelope}}$$

velocity of envelope $= \frac{1}{(1/2)} = 2\text{ m/s}$ phase velocity $= \frac{9}{(9/2)} = 9\left(\frac{2}{9}\right) = 2\text{ m/s}$

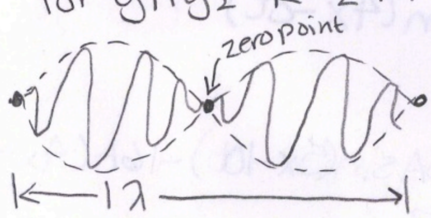
$$y_1+y_3 = A\sin(5x-10t) + A\sin(4x-9t) = 2A\,\underbrace{\sin\left(\tfrac{9}{2}x - \tfrac{19}{2}t\right)}_{\text{high frequency wave}}\,\underbrace{\cos\left(\tfrac{1}{2}x - \tfrac{1}{2}t\right)}_{\text{envelope}}$$

group velocity $= \frac{\frac{1}{2}}{\frac{1}{2}} = 1\text{ m/s}$ phase velocity $= \left(\frac{19}{2}\right)\left(\frac{2}{9}\right) = \frac{19}{9}\text{ m/s}$

for y_1+y_3 the phase velocity is nearly twice the group velocity!

What is the distance between points of zero amplitude in the combined disturbance? What is the frequency of the zero amplitude point (called the beat frequency)?

For this, look at the envelope term,
for y_1+y_2 $k = \frac{1}{2}\text{ m}^{-1}$ for this term $k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k} = 4\pi\text{ m}$



distance between zero points is half a wavelength $= 2\pi\text{ m}$

For y_1+y_3 the answer is the same.

For y_1+y_2 the angular frequency of the envelope is 1 s^{-1}

$\omega = 2\pi\nu$ $\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\text{ s}^{-1}$ and there are 2 beats per envelope wavelength

$$\nu_{beat} = 2\frac{1}{2\pi}\text{ s}^{-1} = \frac{1}{\pi}\text{ s}^{-1}$$

for y_1+y_3 is the envelope has $\omega = \frac{1}{2}\text{ s}^{-1}$ $\nu = \frac{1}{4\pi}\text{ s}^{-1}$

beat frequency $= 2\frac{1}{4\pi}\text{ s}^{-1} = \frac{1}{2\pi}\text{ s}^{-1}$

If you rewrite the original component waves in terms of ν instead of ω, you will see that $\nu_{beat} = |\nu_1 - \nu_2|$