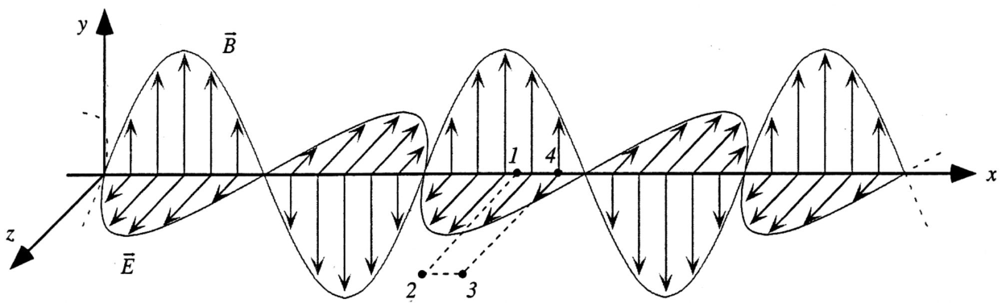
Name:________________________

Maxwell's equations and electromagnetic waves

I. Representations of electromagnetic waves

A. Shown below are mathematical and pictorial representations of an electromagnetic plane wave propagating through empty space. The electric field is parallel to the z-axis; the magnetic field is parallel to the y-axis. ($\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors along the $+x$, $+y$, and $+z$-directions.)

$$\vec{E}(x, y, z, t) = E_o \sin(kx + \omega t)\,\hat{z} \qquad \vec{B}(x, y, z, t) = B_o \sin(kx + \omega t)\,\hat{y}$$



1. In which direction is the wave propagating? Explain how you can tell from the expressions for the electric field and magnetic field.

   It is propagating in the $-\hat{x}$ direction; we know this because the argument of the sine function is $(kx+\omega t)$.

   Is the wave transverse or longitudinal? Explain in terms of the quantities that are oscillating. It is transverse, because the quantities that are oscillating ($\vec{E}$ and $\vec{B}$) are oscillating perpendicular to the direction of motion.

2. The points *1–4* in the diagram above lie in the x–z plane.

   For the instant shown, rank these points according to the magnitude of the *electric field*. If the electric field is zero at any point, state that explicitly.

   1 and 2 are the same, and 3 and 4 are the same. $1,2 > 3,4$. Note that the vectors only show the magnitude of the field and not the extent of the field.

   Is your ranking consistent with the mathematical expression for the electric field shown above? If not, resolve any inconsistencies. (For example, how, if at all, does changing the value of z affect the value of $\vec{E}(x, y, z, t)$?)

   The value of the electric field only depends on x and t so for any given instant in time, 1=2 and 3=4 because they have the same x value

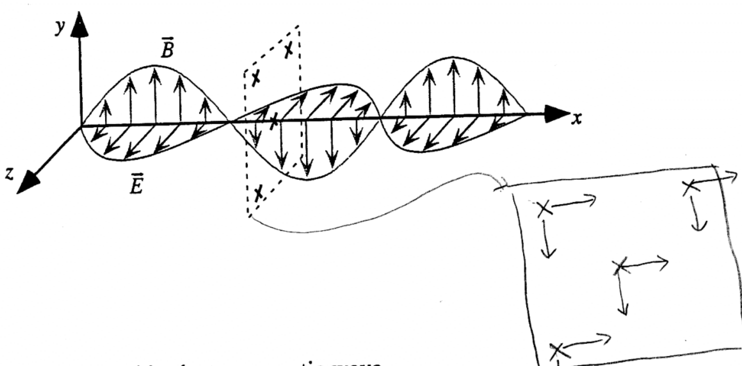
   For the instant shown, rank points *1–4* according to the magnitude of the *magnetic field*. Check that your ranking is consistent with the expression for the magnetic field, $\vec{B}(x, y, z, t)$, above.

   1=2 and 3=4 and $1,2>3,4$

   same argument, the magnitude of $\vec{B}$ only depends on x

In the diagram at right, the four points labeled "x" are all located in a plane parallel to the y–z plane. One of the labeled points is located on the x-axis.

On the diagram, sketch vectors to show the direction and relative magnitude of the electric field at the labeled points.



They are all the same because they are in a plane of constant x.

Justify the use of the term *plane wave* for this electromagnetic wave.

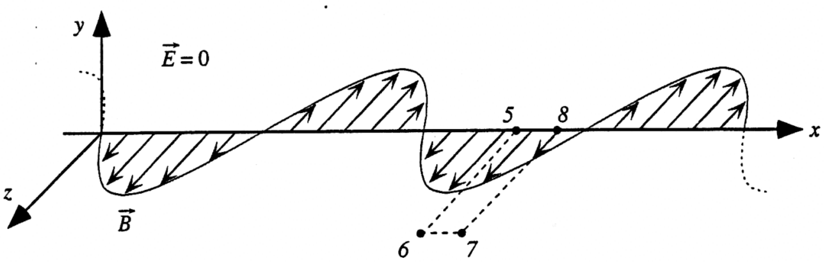
The electic and magnetic fields are constant in any plane transverse to the direction of propagation.

In a region that does not contain current-carrying wires, the magnetic field is found to be $\vec{B}\,(x, y, z, t) = B_o \sin(kx - \omega t)\,\hat{z}$.

Show that in such a region it is not be possible to have an electric field that is equal to *zero* for all x and t. (*Hint:* Set $i_{\text{encl}} = 0$ in Ampère's law: 0 because no current carrying wires

$$\oint \vec{B}\cdot d\vec{l} = \cancel{\mu_o i_{\text{encl}}} + \mu_o \varepsilon_o \frac{d\Phi_E}{dt} \Rightarrow \oint \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

and consider the quantity $\oint \vec{B}\cdot d\vec{l}$ evaluated around the loop $5 \to 6 \to 7 \to 8 \to 5$ in the x–z plane, shown below.)



$$B_z(x_{56})\Delta z + \cancel{B_z(x_{67})\Delta x}^{\,0} + \left(-B_z(x_{78})\right)\Delta z + \cancel{B_z(x_{85})\Delta x}^{\,0}$$

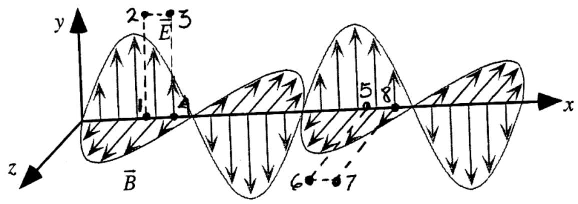
note that we get a negative here because the direction of our path opposes the magnetic field.

$$\left(B_z(x_{56}) - B_z(x_{78})\right)\Delta z = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$B_z(x_{56}) \neq B_z(x_{78}) \Rightarrow B_z(x_{56}) - B_z(x_{78}) \neq 0$$

so $\frac{d\Phi_E}{dt} \neq 0 \Rightarrow$ we must have a changing electric flux.

Consider the following electromagnetic wave:



$$\vec{E}(x,y,z,t) = E_0\sin(kx-\omega t)\hat{y}$$
$$\vec{B}(x,y,z,t) = B_0\sin(kx-\omega t)\hat{z}$$

In class yesterday, we applied Faraday's law $\oint_C \vec{E}\cdot d\vec{\ell} = -\int_S \frac{\partial B_n}{\partial t}dA$

to the curve/surface ~~1-2-3-4~~ 4321-4 and found that: $\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$ $\boxed{A}$

We assumed that the curve C had sides of length Δx and Δy.

Following a similar procedure apply Ampere's law: $\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I + \mu_0\varepsilon_0\int_S \frac{\partial E_n}{\partial t}dA$

to curve ~~5-6-7-8~~ 86758 and show that: $\frac{\partial B_z}{dx} = -\mu_0\varepsilon_0\frac{\partial E_y}{dt}$ $\boxed{B}$

Hint: for an imaginary surface bounded by a closed loop, it is customary to use the right hand rule to determine the direction of the normal that is bounded by that surface. Pay close attention to signs, and the direction of the magnetic field along that loop.

From previous exercise $\oint_C \vec{B}\cdot d\vec{\ell} = \{B_z(x_{87}) - B_z(x_{56})\}\Delta z$
$\approx \frac{\partial B_z}{\partial x}\Delta x\,\Delta z$

now evaluate $\int_S \frac{dE_n}{dt}dA = \frac{d}{dt}\int_S \vec{E}\cdot\hat{n}\,dA$

$\frac{\partial}{\partial t}\int_S \vec{E}\cdot\hat{n}\,dA = -\frac{\partial E_y}{\partial t}\Delta x\,\Delta z$

$\Rightarrow \frac{\partial B_z}{\partial x} = -\mu_0\varepsilon_0\frac{\partial E_y}{\partial t}$

The surface is the surface bounded by 5-6-7-8-5 and the direction of the normal is found by the rhr. The direction of the normal is -y

Take the partial derivative of both sides of equation A with respect to x, then interchange the order of the space and time derivatives on the right:

$$\frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) \qquad \frac{\partial^2 E_x}{\partial x^2} = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right)$$

Now substitute the expression for dB_z/dx into the resulting equation to eliminate B, what do you get? What is the speed of light?

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial t}(\mu_0\varepsilon_0)\frac{\partial E_y}{\partial t} \Rightarrow \frac{\partial^2 E}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial E_y}{\partial t^2}$$

this is the wave eqn with $\frac{1}{v^2} = \mu_0\varepsilon_0$

$v = 1/\sqrt{\mu_0\varepsilon_0}$

If you have time, apply a similar procedure to eliminate E from the equation. Do you get a similar result?

$$\frac{\partial}{\partial t}\left(\frac{\partial E_y}{\partial x}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial t}\right) \text{ (from Eq A)} \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial t}\right) = -\frac{\partial^2 B_z}{\partial t^2}$$

from $\boxed{B}$ $\frac{\partial E_y}{\partial t} = -\frac{1}{\mu_0\varepsilon_0}\frac{\partial B_z}{\partial x}$

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{1}{\mu_0\varepsilon_0}\frac{\partial B_z}{\partial x}\right) = -\frac{\partial^2 B_z}{\partial t^2} \Rightarrow \frac{\partial^2 B_z}{\partial x^2} = \mu_0\varepsilon_0\frac{\partial^2 B_z}{\partial t^2}$$