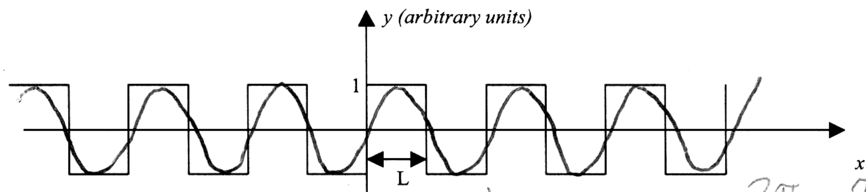
Fourier Analysis and the Uncertainty Principle

I. Fourier Series for a Periodic Function

A. Finding appropriate functions

1. Consider the square wave shown to the right.

   Sketch a single sine or cosine curve that best approximates the square wave.



$|\leftarrow \lambda \rightarrow|$ $\lambda = 2L$ $k = \frac{2\pi}{\lambda} = \frac{\pi}{L}$

   Write a mathematical representation of the function you have sketched. Explain how you came up with your answer. $y = \sin\left(\frac{\pi x}{L}\right)$ I chose a sine because it is an odd function. I choose a wavelength of 2L because the function repeated every x=2L.

B. Exploring the spreadsheet

1. Open up the spreadsheet ***Periodic Square Wave*** on the desktop. The sheet lets you add trigonometric functions of the form *Asin(kx)* and *Bcos(kx)*. In the spreadsheet, $k = {}^{n\pi}/_{L}$.

   Note that the value of *k* is fixed for a given *n*, but the values of *n* and *A* can be varied.

   Find the value of *k* and the value of *n* for the function you obtained in part A.

   $k = \pi/L$ therefore if $k = n\pi/L$, then $n=1$

   In the first function of the spreadsheet, enter the parameters of the function in part A.

2. Consider the blue line in the second graph (if it is not visible, scroll the screen down).

   Describe what the blue line represents in the second graph. Explain how you know. *(Hint: it is given the name "difference.")* It is the difference between the square wave and the sine wave. I know this because its value is zero where the curves touch.

   The number to the right of the second graph gives the value of Δ^2, the area of the difference curve squared. What information does Δ^2 give you about your choice for the function describing the square wave? You may want to try different parameters for the function to see how Δ^2 changes. For my chose $\Delta^2 = 1.38308$

   If I play around with the amplitude, I can get the difference to less than 1.16 with an amplitude of 1.27

Find the best possible value for n and A in your first function. Explain how you used Δ^2 to arrive at your answer.

$\Delta^2 \approx 1.16$ for $A = 1.27 - 1.28$ I arrived at this by playing with A and looking at the resulting Δ^2 (honestly!), but note that this value is very close to the one we calculated in class ($\frac{4}{\pi}$) for the n=1 term of the Fourier series.

C. *Adding more terms to get a better fit*

1. Consider the second graph on the spreadsheet. What curve could you add to the black curve in order to get the square wave? We want something that goes through zero at the x-values where the square wave is zero, but has a maximum amplitude where the difference is greatest

   How could you best approximate the curve you describe with a sine or cosine curve? Explain. I need a sine wave that has a frequency that is an integer multiple of my original wave, so the function is still zero where it needs to be, but there are maxima on either side of the original sine peak.

   Determine values of n and A of your second function. Explain how you arrived at your answer. I get n=3, that makes the wavelength 3L/2, and puts an additional peak on either side of the original peak. If I leave A=1, my difference Δ^2 gets bigger!! Playing with A, I find a best fit with a between 0.4 and 0.45.

2. Use the spreadsheet to add a third trigonometric function to better approximate the square wave. Explain how you optimized the values of A and n of the third function using the difference curve and Δ^2. I get n=5, from the previous step I realize that each term is an ever finer correction, and I guess that A must be less than the ~0.4 I got above. I find 0.25 works well.

D. *Generalizing the Fourier Series*

1. What values of n would you expect for additional terms that describe the square wave on page 1? Describe the general pattern for the wave number, k.

   I expect additional terms to be odd, so
   $k = \frac{\pi}{L}, \frac{3\pi}{L}, \frac{5\pi}{L}, \ldots$

   Suppose the pattern of successive approximations continues. Write the equation for an infinite series that describes the periodic square wave shown on page 1. (For now leave the coefficients undetermined.)

   $y = A \sin\frac{\pi}{L} + A' \sin\frac{3\pi}{L} + A'' \sin\frac{5\pi}{L} + \ldots$

2. The general Fourier series describing an arbitrary function is given as

$$f(x) = \frac{a_o}{2} + \sum_1^{\infty} \{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \}$$

Compare the equation with the series you came up with in the previous question. State the similarities and differences. In this particular example $a_n = 0$ because we don't need the cosine function. Also, we don't need the first term, which corresponds to a general offset.

The values of a_n and b_n can be obtained from the orthogonality conditions for trigonometric functions. We get $a_n = \frac{1}{L}\int_{-L}^{L} f(k)\cos\left(\frac{n\pi k}{L}\right) dk$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(k)\sin\left(\frac{n\pi k}{L}\right) dk$.

II. Localizing an electron using a Fourier Series

A. Understanding Localization

1. Consider an electron in a plane wave state with energy, E, and momentum, p.

Find the angular frequency, ω, in terms of E, and the wave number, k, in terms of p.

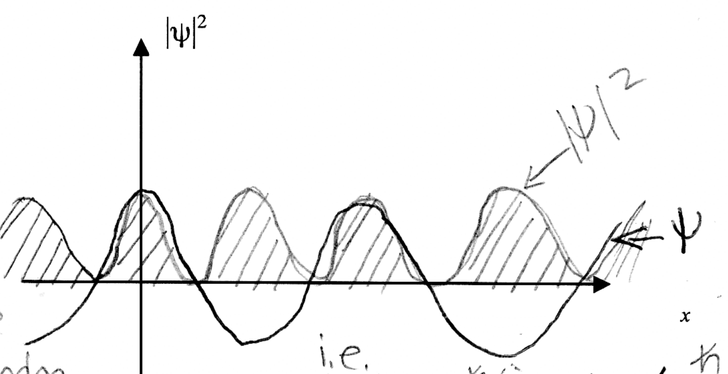
$E = hf \quad \omega = 2\pi f \Rightarrow E = h\frac{\omega}{2\pi} = \hbar\omega \quad \omega = E/\hbar$

deBroglie wavelength $\lambda = h/p \quad k = \frac{2\pi}{\lambda} \Rightarrow k = 2\pi\left(\frac{p}{h}\right) = \frac{p}{\hbar}$

Write out the wave function for the electron in terms of ω and k.

$\psi = A\cos(kx - \omega t)$

In the space to the right, sketch the probability density for this electron.



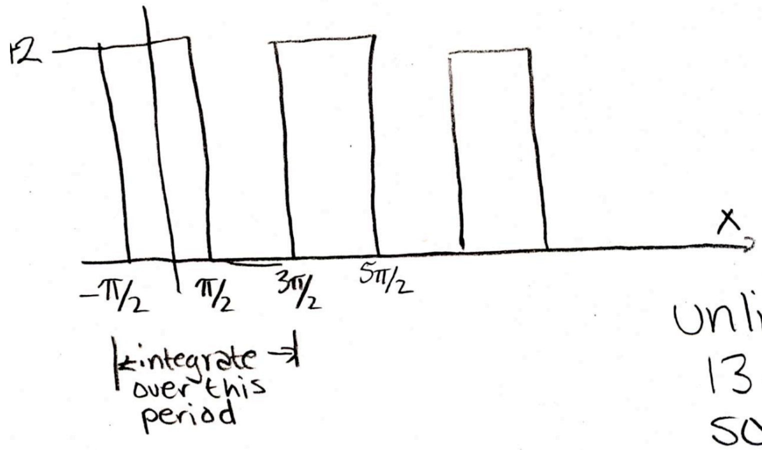
Based on the probability density, what can you say about the position of an electron with energy, E, and momentum, p?

The probability density allows the electron to be anywhere between $-\infty$ and ∞, thus E and p are known precisely. i.e. $(\Delta p \Delta x \leq \hbar/2 \therefore \Delta p \leq \frac{\hbar}{2\Delta x} = \frac{\hbar}{2(\infty)} = 0)$

Would you say the electron with energy, E, and momentum, p, is localized in space? What conditions have to be satisfied for localization? Explain.

This electron is not localized. Using $\Delta p \leq \frac{\hbar}{2\Delta x}$, Δp would be very large if Δx were very small. Localization would introduce an uncertainty in the momentum, and therefore the energy.

our new square wave



note that the average of the function is not zero as in Figure 13.4 but is 1

$a_0 = 1$

Unlike the wave in Figure 13.4, this is an even function so $b_n = 0$ $a_n \neq 0$

To find the a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 2\cos nx\,dx + \cancel{\frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(0)\cos nx\,dx} \to 0$$

$$= \frac{2}{n\pi}\sin nx\,dx\Big|_{-\pi/2}^{\pi/2} = \frac{2}{n\pi}\left(2\sin\left(n\frac{\pi}{2}\right)\right) = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

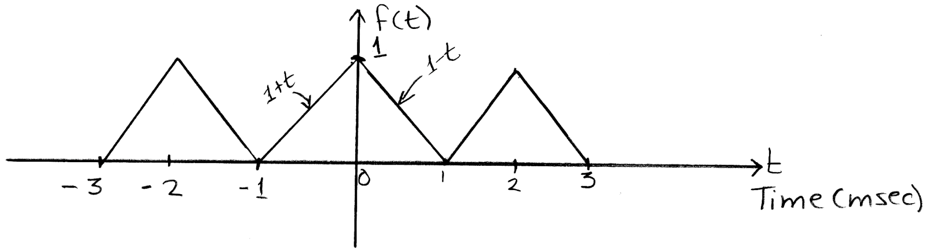
for even n, we get sine of integer multiples of π so a_n for n even $= 0$

for $n=1$ we get $\frac{4}{\pi}$, for $n=3$ we get $\frac{4}{3\pi}\sin\left(\frac{3\pi}{2}\right) = \frac{-4}{3\pi}$

for $n=5$ we get $\frac{4}{5\pi}$ and so on... (every other odd term will be negative)

$$\therefore f(x) = 1 + \frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \dots\right)$$

3.



Note that the wave is given in time coordinates, not space coordinates (it's given in terms of t not x), so evaluate the coefficients using the equations in Section 13.4, which are given in terms of f(t).

a_0, the average, is obviously 0.5 (refer to the picture).

It is an even function (it's symmetric about 0), so referring to equation (13.5) we see that we only need to evaluate the a_n (cosine terms). The coefficient of the sine terms are all 0. ($b_n = 0$)

so we use $a_n = \frac{2}{T}\int_0^T f(t) \cos n\omega_0 t \, dt$

from the picture we see that $T = 2$ (in msec)

angular frequency $\omega_0 = \frac{2\pi}{T}$ (fundamental frequency)

we can integrate over any period (interval over which the function repeats)

I choose the interval $-1 < t < 1$

but you could just as easily choose $1 < t < 3$

or $-3 < t < -1$

over the interval $-1<t<1$, the function is discontinuas, so we have to split the interval into two parts

$-1<t<0 \quad f(t)=1+t$

$0<t<1 \quad f(t)=1-t$

$$a_n=\frac{2}{T}\left[\int_{-1}^{0}(1+t)\cos n\omega_0 t\,dt+\int_{0}^{1}(1-t)\cos n\omega_0 t\,dt\right.$$

$$=\frac{2}{T}\left[\int_{-1}^{0}\cos n\omega_0 t\,dt+\int_{-1}^{0}t\cos n\omega_0 t\,dt+\right.$$

$$\left.\int_{0}^{1}\cos n\omega_0 t\,dt-\int_{0}^{1}t\cos n\omega_0 t\,dt\right]$$

$$=\frac{2}{T}\left[\underbrace{\int_{-1}^{1}\cos n\omega_0 t\,dt}_{\text{combining the first + third integrals}}+\underbrace{2\int_{-1}^{0}t\cos n\omega_0 t\,dt}_{\text{combining the second and fourth integrals}}\right]$$

The first integral is

$$\int_{-1}^{1}\cos n\omega_0 t\,dt=\frac{1}{n\omega_0}\sin n\omega_0 t\Big|_{-1}^{1}=0$$

For the second integral, you must integrate by parts'

$$\int f\frac{dg}{dt}dt=fg-\int\frac{df}{dt}g\,dt \quad \text{(Appendix B.2)}$$

$\int_{-1}^{1} t\cos n\omega_o t\,dt$ let $f=t$ $\frac{df}{dt}=1$

$\frac{dg}{dt}=\cos n\omega_o t$ $g=\frac{1}{n\omega_o}\sin n\omega_o t$

$$=\underbrace{\frac{t}{n\omega_o}\sin n\omega_o t\Big|_{-1}^{0}}_{=0}-\frac{1}{n\omega_o}\int_{-1}^{0}\sin n\omega_o t\,dt$$

$$=\frac{1}{(n\omega_o)^2}\cos n\omega_o t\Big|_{-1}^{0} \qquad \omega_o=\frac{2\pi}{T}=\frac{2\pi}{2}=\pi$$

$$=\frac{1}{n^2\pi^2}\left[\cos 0-\underbrace{\cos(-n\pi)}_{\substack{=1 \text{ for even } n\\ =-1 \text{ for odd } n}}\right]=\begin{cases}0 & n \text{ even}\\ 2 & n \text{ odd}\end{cases}$$

plugging this back into expression for a_n

$$a_n=\frac{2}{T}\left[0+2\frac{2}{n^2\pi^2}\right]=\frac{4}{n^2\pi^2} \quad (\text{for odd } n)$$

even $a_n=0$

plug in $T=2$ 1

Plugging this into the series

$$f(t)=0.5+\frac{4}{\pi^2}\cos\omega_o t+\frac{4}{9\pi^2}\cos 3\omega_o t$$

$$+\frac{4}{25\pi^2}\cos 5\omega_o t+\cdots$$

4. This diagram is the same type of diagram shown in Figure 13.6 in your book. It plots the amplitude against the frequency for each term in the series.

a) So reading from the plot, the amplitude, a_n, of the term corresponding to $\nu = 9.83 \times 10^5$ Hz is 1.0

Therefore, that term in the series is

$$a_n \cos \omega t = 1.0 \cos[2\pi(9.80 \times 10^5)t]$$

the a_n of the term at $\nu = 9.804 \times 10^5$ Hz is

$a_n = 0.4$ so that term is $0.4\cos[2\pi(9.804 \times 10^5)t]$

and so on...

Therefore

$$f(t) = 0.45\cos[2\pi(9.775 \times 10^5)t] + 0.2\cos[2\pi(9.786 \times 10^5)t + 0.3\cos[2\pi(9.790 \times 10^5)t] + 0.35\cos[2\pi(9.793 \times 10^5)t] + 0.4\cos[2\pi(9.796 \times 10^5)t] + 1.0\cos[2\pi(9.800 \times 10^5)t] + 0.4\cos[2\pi(9.804 \times 10^5)t] + 0.35\cos[2\pi(9.807 \times 10^5)t] + 0.3\cos[2\pi(9.810 \times 10^5)t] + 0.20\cos[2\pi(9.814 \times 10^5)t] + 0.45\cos[2\pi(9.83 \times 10^5)t]$$

b) This problem is much like Chapter 13 Example 1 in your book.
The first part of the signal comes at 9.775×10^5 Hz and ends at 9.825×10^5 Hz
So the frequency span is 0.5×10^5 Hz.
So you would have to "tune in" for $T = \frac{1}{0.5 \times 10^5} s = 0.2 ms$ to capture the whole waveform.

5. Look at figure 13.11.
The period $T = 1 \mu sec$ $\quad \omega_0 = \frac{2\pi}{1 \mu s} = \frac{2\pi}{1 \times 10^{-6} s} = 2\pi \times 10^6$ Hz
This is the central frequency corresponding to ω_0 in the figure.
The pulse width is $10 \times 10^{+6} s = 1 \times 10^{+5} s$
So the bandwidth is $\frac{2\pi}{1 \times 10^5}$ Hz $= 2\pi \times 10^{-5} s^{-1}$
(used equation 13.10)

This is a beat pattern. Thus it it results from adding 2 waves of the same amplitude.
From the graph we see that the beat period $T_{beat} = 0.005s$

$\Rightarrow \nu_1 - \nu_2 = \frac{1}{0.005s} = 200Hz$

The "phase period" is $\frac{0.0044s}{9s} = 0.00049s \Rightarrow \frac{\nu_1 + \nu_2}{2} = \frac{1}{0.00049s}$

$\Rightarrow \nu_1 + \nu_2 = 2/(0.00049s) = 4090Hz$

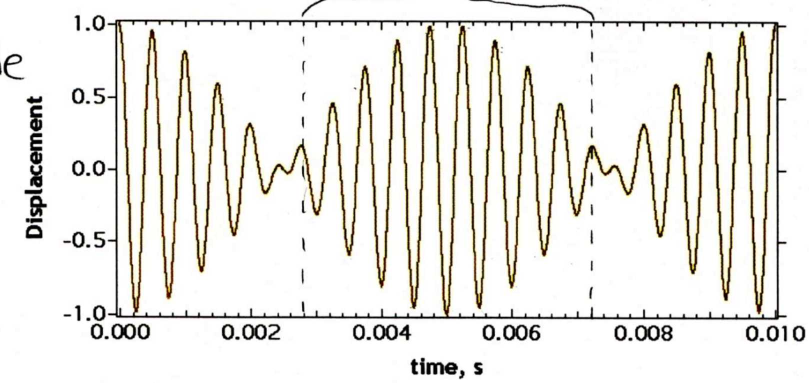
solve for ν_1:

$$\begin{array}{r} \nu_1 - \nu_2 = 200Hz \\ \nu_1 + \nu_2 = 4090Hz \\ \hline 2\nu_1 = 4290Hz \\ \nu_1 = 2145Hz \end{array}$$

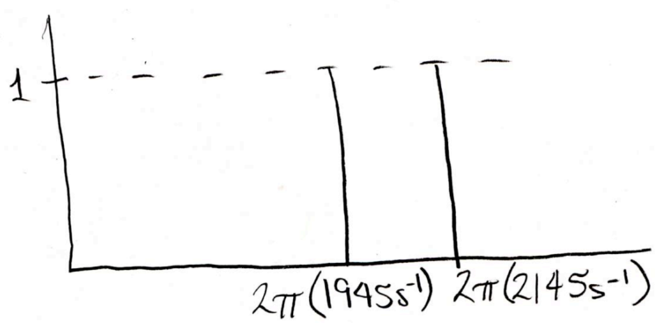
$$\begin{aligned} \nu_1 - \nu_2 &= 200Hz \\ \nu_2 &= 2145Hz - 200Hz \\ &= 1945Hz \end{aligned}$$

9 "phase wavelengths" in $0.0072s - 0.0028s = 0.0044s$

The amplitude of each wave is 1.0.



The spectrum:



The Fourier series:

$f(x) = \cos[2\pi(1945s^{-1})t] + \cos[2\pi(2145s^{-1})t]$