1.

$\omega = 10^3 k - 3 \times 10^{-5} k^3$

(a) See plot in separate file. The maximum occurs where $dw/dk = 0$ which can be found using the derivative taken in part (c) to be at $k \approx \pm 3.3 \times 10^3$.

(b) It is dispersive because

$$\frac{\omega}{k} \neq \frac{d\omega}{dk} \tag{1}$$

In words, the pase and group velocities are not equal.

(c) Find phase and group velocities at $k = 1 \times 10^3$ rad/m.

$$v_{ph} = \frac{\omega}{k} = 10^3 - 3 \times 10^{-5} k^2 = 10^3 - (3 \times 10^{-5})(1 \times 10^3)^2 \tag{2}$$
$$= (1000 - 30)\text{m/s} = 970\text{m/s} \tag{3}$$

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(10^3 k - 3 \times 10^{-5} k^3\right) = 10^3 - 9 \times 10^{-5} k^2 \tag{4}$$
$$= 10^3 - (9 \times 10^{-5})(1 \times 10^3)^2 \tag{5}$$
$$= (1000 - 90)\text{m/s} = 910\text{m/s} \tag{6}$$

2.

Two sinusoidal waves:

$$\xi_1 = 2\sin(5x - 1500t), \xi_2 = 2\sin(5.1x - 1530t) \tag{7}$$

(a) Using equation (2.39) from your book, with $k_1 = 5$, $\omega_1 = 1500$, $k_2 = 5.1$ and $\omega_2 + 1530$ you get

$$f(x,t) = 2A\sin\left[\frac{(k_1 + k_2)x - (\omega_1 + \omega_2)t}{2}\right]\cos\left[\frac{(k_1 - k_2)x - (\omega_1 - \omega_2)t}{2}\right] \tag{8}$$

(b) The beat frequency is given by the lower frequency term, which is the cosine term in the equation above. That determines the "envelope" in Figure 2.18. Note that it repeats every π (not 2π).

$$\frac{(\omega_1 - \omega_2)}{2} = \pi\nu_{\text{beat}} \tag{9}$$

Plugging in ω_1 and ω_2, you get 4.8 Hz.

(c) The beat wavelength can be found in several ways. One way is to note that the velocity $v = \omega/k$ which is 1500/5=300 m/s. Then plug the beat frequency into $\lambda = v/\nu$= 63 m. The other way is to get it from the definition of k ($k = \lambda/2\pi$) using the k terms in the envelope function.

(d) Find the phase and group velocities. Since we added two waves with equal velocities ($\omega_1/k_1 = \omega_2/k_2$)=300 m/s, the phase velocity and group velocity of the combined wave will be the same ($v_p h = v_g$) and equal to 300 m/s.

3.

$$E = Ae^{i(kx-\omega t)}$$

$$\frac{\partial^2 E}{\partial t^2} = A(-i\omega)^2 e^{i(kx-\omega t)}$$

$$\frac{\partial^2 E}{\partial x^2} = A(ik)^2 e^{i(kx-\omega t)}$$

plugging this into the equation:

$$\frac{\partial^2 E}{\partial t^2} + \omega_p^2 E = c^2 \frac{\partial^2 E}{\partial x^2}$$

$$A(-i\omega^2)^2 e^{i(kx-\omega t)} + \omega_p^2 Ae^{i(kx-\omega t)} = c^2 (ik)^2 Ae^{i(kx-\omega t)}$$

$$-\omega^2 + \omega_p^2 = -c^2k^2$$

$$\omega^2 = \omega_p^2 + c^2k^2$$ dispersion relation

4.

Show that the differential equation that yields the dipersion relation $\omega^2 = gk$ is given by

$$\frac{\partial^2 \xi}{\partial t^2} = ig\frac{\partial \xi}{\partial x} \tag{10}$$

An equation of motion is an equation that describes how the acceleration evolves over time, so we need to find the acceleration.

Assume a solution of the type $f(x,t) = Ae^{i(kx-\omega t)}$.

$$\frac{\partial^2 \xi}{\partial t^2} = -\omega^2 Ae^{i(kx-wt)} = -\omega^2 f(x,t) \tag{11}$$

Plugging in the dispersion relation this becomes $-gkf(x,t)$. We need a k on the right side of the equation then, and taking the first derivative of our solution with respect to x yields a factor of k.

$$\frac{\partial \xi}{\partial x} = ikAe^{i(kx-wt)} = ikf(x,t) \tag{12}$$

So hypothesize that the equation has the form

$$\frac{\partial^2 \xi}{\partial t^2} = c\frac{\partial \xi}{\partial x} \tag{13}$$

5.

a) $C_w = \sqrt{\frac{T}{\rho_e}} = \sqrt{\frac{50N}{0.1 Kg/m}} = 22.4 m/s$

b) $f = \frac{1}{T} = \frac{1}{0.1s}$ $\quad \lambda = \frac{V}{f} = (22.4 m/s)(0.1s) = 2.24m$

c) The wave is moving in the +x direction so it has the form $y(x,t) = A\sin(Kx - \omega t + \phi)$ ← phase shift

where $A = 0.02m$

$K = \frac{2\pi}{\lambda} = \frac{2\pi}{2.24m} = 2.80 m^{-1}$ $\quad$ at $x=0$ $t=0$ $\quad y = A\sin\phi$

$\omega = 2\pi f = 62.8 s^{-1}$

now find phase shift ϕ

$\frac{dy}{dt} = -\omega A \cos(Kx - \omega t + \phi)$

at $x=0$ $t=0$

$\frac{dy}{dt} = -\omega A \cos\phi$ $\quad$ velocity is negative as $t=0$ $x=0$ so $\cos\phi$ is positive

from the initial condition that $y = 0.01m$

$\therefore \sin\phi = \frac{1}{2} \Rightarrow \phi = 30$ degrees $= \frac{\pi}{6} = 0.52 rad$

$y(x,t) = 0.02\sin(2.80x - 62.8t + \frac{\pi}{6})$

6.

H&L Ch4 #4

$$\mathcal{E}(x,t) = \mathcal{E}_o e^{-(x-c_\omega t)^2/a^2}$$

a) The kinetic energy density can be found from Eq 4.17 with ρ_e substituted for m: $KE|_{density} = \frac{1}{2}\rho_e\left(\frac{\partial \mathcal{E}}{\partial t}\right)^2$

$$\frac{\partial \mathcal{E}}{\partial t} = 2c_\omega \frac{(x-c_\omega t)^2}{a^2}\mathcal{E}_o e^{-(x-c_\omega t)^2/a^2}$$

$$\therefore KE = \int_{-\infty}^{\infty} \frac{1}{2}\rho_e \left[\frac{(x-c_\omega t)^2}{a^2} 2c_\omega \mathcal{E}_o e^{-(x-c_\omega t)^2/a^2}\right]^2 dx$$

$$= 2\rho_e \left(\frac{\mathcal{E}_o c_\omega}{a^2}\right)^2 \int_{-\infty}^{\infty} (x-c_\omega t)^2 e^{-2(x-c_\omega t)^2/a^2} dx$$

$$= 2\rho_e \left(\frac{\mathcal{E}_o c_\omega}{a^2}\right)^2 \int_{-\infty}^{\infty} u^2 e^{-2u^2/a^2} du \quad \text{where } u = x - c_\omega t$$

looking this integral up in an integral table I find $\int_{-\infty}^{\infty} x^2 e^{-cx^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{c^3}}$

in my integral, the constant $c = -2/a^2$

$$= 2\rho_e \left(\frac{\mathcal{E}_o c_\omega}{a^2}\right)^2 \frac{1}{2}\sqrt{\frac{\pi a^6}{8}}$$

simplifying:

$$= \sqrt{\frac{\pi}{2}}\, \rho_e (\mathcal{E}_o c_\omega)^2 / 2a$$

b) see discussion on p. 63 of your book $KE = PE$

c) see equation 4.43 $\quad$ momentum density $= \frac{\text{energy density}}{c_\omega}$

Total Energy $= KE + PE = \sqrt{\frac{\pi}{2}}\, \rho_e (\mathcal{E}_o c_\omega)^2 / a$

momentum $= \sqrt{\frac{\pi}{2}}\, \rho_e\, \mathcal{E}_o^2 c_\omega / a$

7.

(a) Find the velocity of the transverse waves.

We can use equation (4.44), which we derived in class a few weeks ago. (see your notes, in class we used μ instead of ρ for the density)

$$v = \sqrt{\frac{T}{\rho_l}} \tag{16}$$

With $T = 10N$ and ρ_l=20 g/m=0.020 kg/m (note that I changed to SI units), we get 22.4 m/s.

(b) Find the frequency.

The transverse displacement is given as $\xi = 0.01\text{m}\sin(20t)$, and should have the form $\xi(t) = A\sin\omega t$, therefore the angular frequency ω is 20 rad/s. We know the frequency is related to ω by $\nu = \omega/2\pi$. Therefore the frequncy is 3.2 Hz.

(c) Find the wavelength.

We know that the velocity, frequency and wavelength are related by $v = \lambda\nu$ and we have v and ν. Using $\lambda = v/\nu$, you get 7.0 m.

(d) Find the average power.

See the equations in the middle of page 72. The power is the energy density multiplied by the velocity. The average energy density is:

$$E_{av} = \frac{1}{2}\rho_l\omega^2\eta_0^2 = \frac{1}{2}(0.020\text{kg/m})(20\text{rad/s})^2(0.01\text{m})^2 = 4\times10^{-4}\text{Joules} \tag{17}$$

Then the power delivered is vE_{av}=9 mW.