66. resonant frequencies $500 < f < 1600\,kHz$

$L = 1\,\mu H$

what range of capacitances?

$\omega = 2\pi f$ ⇒ the range of frequencies corresponds to angular frequencies

$3141 < \omega < 10{,}053 \times 10^3\ rad/s$

$\omega = \frac{1}{\sqrt{LC}} \qquad C = \frac{1}{\omega^2 L}$

for the low end of the frequency range.

$C = \frac{1}{(3141 \times 10^3/s)^2 (1 \times 10^{-6} H)} = 0.1\ \mu F$

for the high end

$C = \frac{1}{(10{,}053 \times 10^3/s)^2 (1 \times 10^{-6} H)} = 9.9\,nF$

83. $Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} = \sqrt{R^2 + (X_L - X_C)^2}$

a) for an LR circuit $X_C = 0$

$Z = \sqrt{R^2 + \omega^2 L^2}$

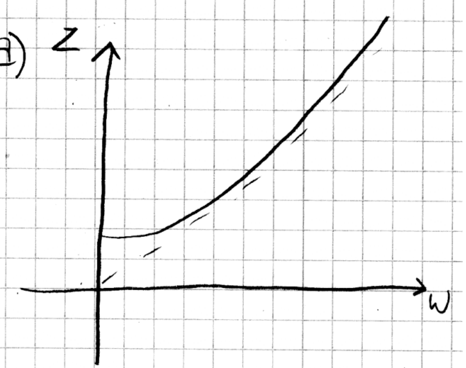
rearranging

$Z^2 = R^2 + \omega^2 L^2$

$Z^2 - \omega^2 L^2 = R$

This is the equation of a hyperbola

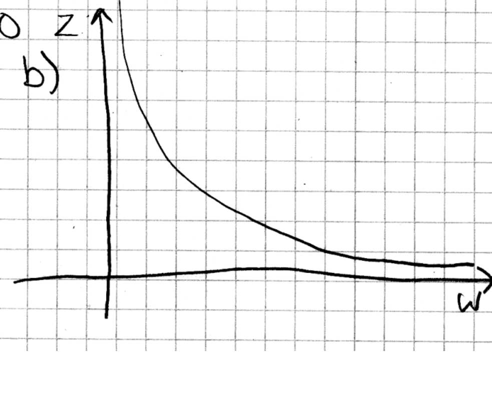
a)



b) for an RC circuit $X_L = 0$

$Z = \sqrt{R^2 + \frac{1}{\omega^2 C^2}}$

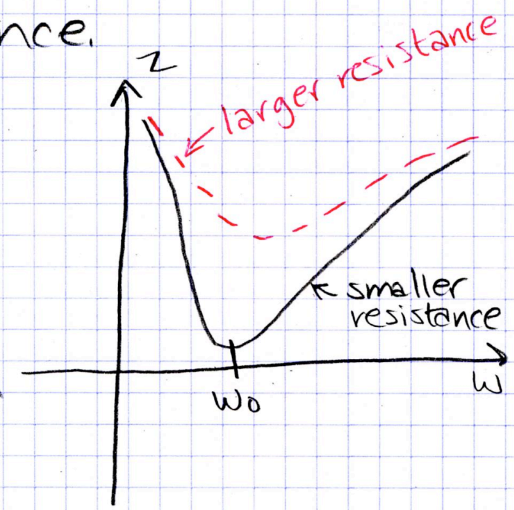
b)



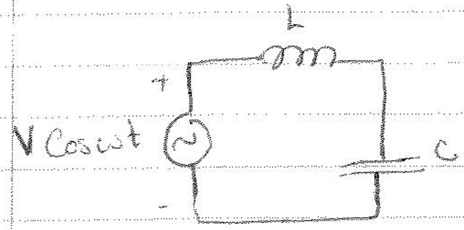
c) For a series RLC circuit, the current is maximum maximum at resonance as shown in Figure 29-20. The impedance is thus a minimum a resonance.

$$Z = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

The smaller the resistance, R, the sharper the impedance minimum will be.



#4) (29.99)



a) Voltage drop across the generator is $-V \cos \omega t$.

where V is the maximum emf $= \mathcal{E}_{max}$

Now voltage drop across inductor is $L \dfrac{d\, i(t)}{dt}$

where $i(t) = \dfrac{d\, Q(t)}{dt}$

$\Rightarrow$ across inductor, the drop is $L \dfrac{d^2 Q(t)}{dt^2}$

The sum of voltage drops in the circuit should

equal 0 $\qquad$ drop across Capacitor

$$\Rightarrow L \frac{d^2 Q(t)}{dt^2} + \underbrace{\frac{Q}{C}}_{\text{drop across Capacitor}} - \mathcal{E}_{max} \cos \omega t = 0$$

$$\longrightarrow L \frac{d^2 Q(t)}{dt^2} + \frac{Q}{C} = \mathcal{E}_{max} \cos \omega t \qquad (1)$$

(b) $Q = Q_{max} \cos \omega t$

$$\Rightarrow \frac{d^2 Q(t)}{dt^2} = -Q_{max} \cos(\omega t) \cdot \omega^2$$

want to show $\Rightarrow$ $-L Q_{max} \cancel{\cos \omega t} \cdot \omega^2 + \frac{1}{C} \cdot Q_{max} \cancel{\cos \omega t} = \mathcal{E}_{max} \cancel{\cos \omega t}$

$$\Rightarrow -L Q_{max} \cdot \omega^2 + \frac{Q_{max}}{C} = \mathcal{E}_{max}$$

$$\Rightarrow Q_{max} \left(-L\omega^2 + \frac{1}{C}\right) = \mathcal{E}_{max} \quad ; \quad C = \frac{1}{L\omega_0^2} \left(\text{from } \omega_0 = \frac{1}{\sqrt{LC}}\right)$$

$$\Rightarrow Q_{max} \left(-L\omega^2 + L\omega_0^2\right) = \mathcal{E}_{max}$$

We were given that $Q_{max} = -\frac{\mathcal{E}_{max}}{L(\omega^2 - \omega_0^2)}$

$$\Rightarrow \frac{-\mathcal{E}_{max}}{\cancel{L(\omega^2 - \omega_0^2)}} \left(\overset{-1}{\cancel{-L\omega^2 + L\omega_0^2}}\right) = \boxed{\mathcal{E}_{max}}$$

(c) $I(t) = \frac{d\,Q(t)}{dt} = \frac{\omega \mathcal{E}_{max}}{L(\omega_0^2 - \omega^2)} \sin \omega t$

Case 1: $\omega < \omega_0$ Current is positive

$$\Rightarrow \sin(\omega t) = \cos(\omega t + 90)$$

$$\Rightarrow I = \frac{\omega \mathcal{E}_{max}}{L(\omega_0^2 - \omega^2)} \cos(\omega t \cancel{- (-\delta)}^{+90^\circ})$$

Case 2: $\omega > \omega_0 \Rightarrow I < 0 \Rightarrow \sin(\omega t) = -\cos(\omega t - 90)$

$\Rightarrow$ We can write $I = \dfrac{\omega \mathcal{E}_{max}}{L|\omega_s^2 - \omega_0^2|} \cos(\omega t - 90^\circ)$

Now, we have to show that $\dfrac{\omega \mathcal{E}_{max}}{L|\omega_s^2 - \omega_0^2|} = \dfrac{\mathcal{E}_{max}}{|X_L - X_C|}$

$X_L - X_C = \omega L - \dfrac{1}{\omega C}$; $C = \dfrac{1}{L\omega_0^2}$

$$\Rightarrow \omega L - \frac{L\omega_0^2}{\omega} = \frac{L}{\omega}\left(\omega^2 - \omega_0^2\right)$$

$$\Rightarrow \frac{\omega}{L}\left(\omega^2 - \omega_0^2\right) = \frac{1}{|X_L - X_C|}$$

$$\Rightarrow I = I_{max} \cos(\omega t - \delta) \text{ ;}$$

where $I_{max} = \dfrac{\omega \mathcal{E}_{max}}{L|\omega^2 - \omega_0^2|} = \boxed{\dfrac{\mathcal{E}_{max}}{|X_L - X_C|}}$

6) (1.11)

$C = 5\mu F$; $V = 1V$; $L = 2\mu H$; $R = 5m\Omega$

a) $\sqrt{L/C} = \sqrt{\frac{2\mu H}{5\mu H}} = 0.632$

$0.632 \gg 5\,mH \Rightarrow$ Condition for weakly damped is satisfied.

b) $U = 1/2\,\frac{q^2}{C}$

$q(t) = q_0 e^{-\gamma t} \cos \omega t \quad \Rightarrow U = \frac{q_0^2}{2C} e^{-2\gamma t} \cos^2 \omega t$

~~$\Rightarrow U = 1/2\,U$ when $q = \frac{q_0}{\sqrt{2}}$~~

$U = 1/2\,U_0 \Rightarrow \frac{U}{U_0} = \frac{q_0^2/2C\, e^{-2\gamma t} \cos^2 \omega t}{q_0^2/2C} = 1/2$

Let $\cos^2 \omega t = 1$ (Amplitude of $q(t)$)

$\Rightarrow 1/2 = e^{-2\gamma t} \Rightarrow \ln(1/2) = -2\gamma t$

$\Rightarrow t = \frac{\ln(1/2)}{-2\gamma} = -\frac{L}{R} \ln(1/2)$, where $\gamma = \frac{R}{2L}$

$\Rightarrow$ ~~$t = \frac{5\mu H}{2\cdot 2\mu H}$~~ $t = -\frac{2\mu H}{5m\Omega} \ln(1/2) = 2.77 \times 10^{-4} s$

$\Rightarrow \boxed{t \approx 0.28\,ms}$

5. $m = 0.2\,kg$ $f = 4\,N\cdot s/m$ $K = 80\,N/m$ mechanical spring system
driving force: $F = F_0 \cos \omega t$ $F_0 = 2N$ $\omega = 30 s^{-1}$

If the motion is decribed by $x = A\cos(\omega t + \delta)$
Equations 14-53 and 14-54 in Tipler and Mosca can be used to find the values of the constants A and δ

14-53 $$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + f^2\omega^2}}$$ where $\omega_0 = \sqrt{\frac{K}{m}}$

$$\omega = \sqrt{\frac{(80\,N/m)}{(0.2\,kg)}} = 20 s^{-1}$$

$$A = \frac{2N}{\sqrt{(0.2\,kg)^2\left[(20s^{-1})^2 - (30s^{-1})^2\right]^2 + (4\,\tfrac{N\cdot s}{m})^2(30s^{-1})^2}}$$

$$= 1.3\,cm$$

$$\tan(-\delta) = \frac{f\omega}{m(\omega_0^2 - \omega^2)} = \frac{(4\,N\cdot s/m)(30s^{-1})}{0.2\,kg\left[(20s^{-1})^2 - (30s^{-1})^2\right]}$$

note the T&M assumes a motion of the fo $x = A\cos(\omega t - \delta)$ so we have to take the negative of the resulting angle

$$\tan(-\delta) = -1.2$$
$$\delta = 50°$$

I didn't actually expect you to be able to derive these equations, but many of you attempted to do just that... and alot of you got pretty close. Congratulations!

For those of you who attempted this, I have included the derivation of these expressions on the following pages.

Solving the differential equation for forced oscillations with damping.

$F = ma$

$$m\frac{d^2x}{dt^2} = \underbrace{-kx}_{\text{spring force}} \underbrace{-f\frac{dx}{dt}}_{\text{viscous damping force}} + \underbrace{F_0 \cos\omega t}_{\text{driving force}}$$

rearranging:

$$\frac{d^2x}{dt^2} + \frac{f}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t$$

let $\frac{f}{m} = \gamma$ and noting that the resonant frequency $\omega_0 = \sqrt{\frac{k}{m}}$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

This is much simpler if we solve this in the complex plane, then take the real parts at the end to describe the motion

noting that $\cos\omega t = Re(e^{i\omega t}) = Re(\cos\omega t + i\sin\omega t)$

$$\frac{d^2z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t} \quad \text{(in the complex plane)}$$

the solution is of the form $z = Ae^{i(\omega t - \delta)}$

where the motion is $x = Re(z) = A\cos(\omega t - \delta)$

$$\frac{dz}{dt} = i\omega Ae^{i(\omega t - \delta)} \qquad \frac{d^2z}{dt^2} = -\omega^2 Ae^{i(\omega t - \delta)}$$

plugging this in to the differential equation:

$$-\omega^2 A e^{i(\omega t-\delta)} + \gamma i\omega A e^{i(\omega t-\delta)} + \omega_0^2 A e^{i(\omega t-\delta)} = \frac{F_0}{m} e^{i\omega t}$$

now multiply both sides of the equation by $e^{-i(\omega t-\delta)}$ (Note that this is equivalent to dividing by $e^{i(\omega t-\delta)}$)

$$-\omega^2 A + i\gamma\omega A + \omega_0^2 = \frac{F_0}{m} e^{i\delta}$$

to separate the real and imaginary parts, write

$$e^{i\delta} = \cos\delta + i\sin\delta$$

$$-\omega^2 A + i\gamma\omega A + \omega_0^2 A = \frac{F_0}{m}\cos\delta + i\frac{F_0}{m}\sin\delta$$

seperately setting the real and imaginary parts equal:

Ⓐ $\gamma\omega A = \frac{F_0}{m}\sin\delta$

Ⓑ $(\omega_0^2 - \omega^2) A = \frac{F_0}{m}\cos\delta$

$\tan\delta$ is given by equation Ⓐ divided by equation Ⓑ

$$\tan\delta = \frac{\gamma\omega}{(\omega_0^2-\omega^2)} = \frac{f\omega}{m(\omega_0^2-\omega^2)}$$

to find A, use Euler's formula! Square Ⓐ and Ⓑ and add them:

$$\gamma^2\omega^2 A^2 = \frac{F_0^2}{m^2}\sin^2\delta$$

$$+(\omega_0^2-\omega^2)^2 A^2 = \frac{F_0^2}{m^2}\cos^2\delta$$

$$\left[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2\right]A^2 = \frac{F_0^2}{m^2}\underbrace{(\sin^2\delta + \cos^2\delta)}_{=1}$$

now solve for A

$$A^2 = \frac{F_o^2}{m^2(\omega_o^2-\omega^2)^2 + m^2\gamma^2\omega^2} \quad \text{but } \gamma = f/m$$

$$A = \frac{F_o}{\sqrt{m^2(\omega_o^2-\omega^2)^2 + f^2\omega^2}}$$

If you insisted on using cosines instead of exponentials.... read on, but it's not pretty.

assume $x = A\cos(\omega t - \delta)$ is the solution

$$\frac{dx}{dt} = -\omega A\sin(\omega t - \delta) \qquad \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t-\delta)$$

plug this into: $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = \frac{F_o}{m}\cos\omega t$

and you get:

$$-\omega^2 A\cos(\omega t-\delta) - \gamma\omega A\sin(\omega t-\delta) + \omega_o^2 A\cos(\omega t-\delta) = \frac{F_o}{m}\cos\omega t$$

Now, we need to write $\cos\omega t$ in terms of $\cos(\omega t-\delta)$ or $\sin(\omega t-\delta)$

So... write $\cos\omega t$ as $\cos[(\omega t-\delta)+\delta]$

Then use the trigonometric identity:

$$\cos(\theta+\alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

to write this as:

$$\cos\omega t = \cos[(\omega t-\delta)+\delta] = \cos(\omega t-\delta)\cos\delta - \sin(\omega t-\delta)\sin\delta$$

Plugging this in for the term to the right of the "=" sign, we get

$$-\omega^2 A\cos(\omega t-\delta)-\omega A\gamma\sin(\omega t-\delta)+\omega_0^2 A\cos(\omega t-\delta)$$
$$=\frac{F_0}{m}\cos(\omega t-\delta)\cos\delta-\frac{F_0}{m}\sin(\omega t-\delta)\sin\delta$$

Note that the terms $\cos\delta$ and $\sin\delta$ are constants; set the terms in $\cos(\omega t-\delta)$ equal and the terms in $\sin(\omega t-\delta)$ equal.

$$-\omega^2 A\cancel{\cos(\omega t-\delta)}+\omega_0^2 A\cancel{\cos(\omega t-\delta)}$$
$$=\frac{F_0}{m}\cancel{\cos(\omega t-\delta)}\cos\delta$$

$$-\omega A\gamma\cancel{\sin(\omega t-\delta)}=-\frac{F_0}{m}\cancel{\sin(\omega t-\delta)}\sin\delta$$

simplifying these equations, you get

B $(\omega_0^2-\omega^2)A=\frac{F_0}{m}\cos\delta$

A $\omega A\gamma=\frac{F_0}{m}\sin\delta$

These are the exact same solution we came up with when solving with complex exponentials, and the remainder of the solution is the same as before!!

You know I wouldn't lie to you when I told you the two methods were equivalent. Wasn't it easier using exponentials??