H&L . Chapter 3 . # 3)

(a) want to show $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$

$e^{ix} = \cos x + i\sin x$

$$\Rightarrow \cos x = \frac{\cos x + \cancel{i\sin x} + \cos x - \cancel{i\sin x}}{2} = \frac{\cancel{2}\cos x}{\cancel{2}} = \cos x$$

We can also prove this using Taylor Expansion of e^{ix} & $\sin(x)$ &

$\cos(x)$.

$$e^{ix} = \sum_{n=0}^{\infty} i^n \frac{x^n}{n!} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

$$\Rightarrow \frac{e^{ix}+e^{-ix}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty} i^n \frac{x^n}{n!} + \sum_{n=0}^{\infty} i^n \frac{(-x)^n}{n!}\right]$$

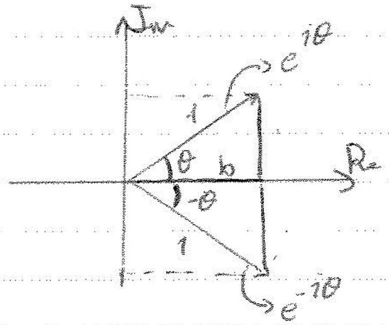
$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left[i^n\frac{x^n}{n!} + i^n\frac{(-x)^n}{n!}\right]\right] = \frac{1}{2}\sum_{n=0}^{\infty} 2 i^n \frac{x^n}{n!} \quad \text{when } n \text{ is even}$$

$\Rightarrow = 0 \quad n$ is odd

$$\Rightarrow = \frac{1}{2}\sum_{n=0}^{\infty} 2(-1)^n \frac{x^n}{n!} = \cos(x)$$

$\Rightarrow$ This is Taylor Series representation of $\cos x$.

Geometric Representation.



$b = \cos\theta \Rightarrow \cos\theta = e^{i\theta}$

~~b is shared~~

~~$b = e^{i\theta}$ & $e^{-i\theta}$~~

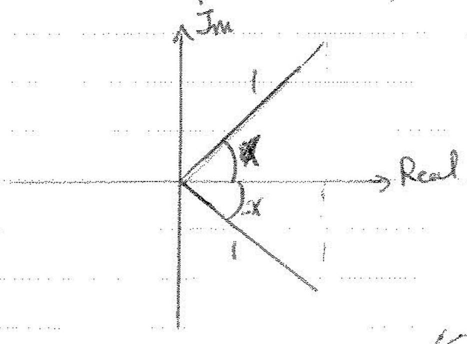
when we add the two, $e^{i\theta}$ & $e^{-i\theta}$, the imaginary parts cancel out and twice the real part remains. The real part is just $\cos\theta$

$\Rightarrow \cos\theta = \frac{\cancel{2} \cdot \text{real part}}{\cancel{2}} = \cos\theta$

(b) $\sin x = \frac{e^{i\cancel{x}} - e^{-i\cancel{x}}}{2i}$

again: $\sin x = \frac{\cos\cancel{x} + i\sin\cancel{x} - \cos\cancel{x} + i\sin\cancel{x}}{2i} = \frac{\cancel{2i}\sin\cancel{x}}{\cancel{2i}} = \sin x$

. Using the same argument as in part a, we can use taylor series expansion to prove $\sin x$.



again. Subtracting e^{-ix} from e^{ix} would result in cancellation of real parts $\Rightarrow$ Twice the imaginary part remains $\Rightarrow \frac{\cancel{2} \cdot \overbrace{\text{Imag. Part}}^{i\sin x}}{\cancel{2i}} = \sin x$

Problem #4)

$$\frac{d}{d\theta}\left(Ae^{i\theta}\right) = iAe^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Rightarrow \frac{d}{d\theta}\left(A\left(\cos\theta + i\sin\theta\right)\right) = A \cdot \frac{d}{d\theta}\left(\cos\theta + i\sin\theta\right)$$

$$= A\left(-\sin\theta + i\cos\theta\right) \quad (1)$$

now;

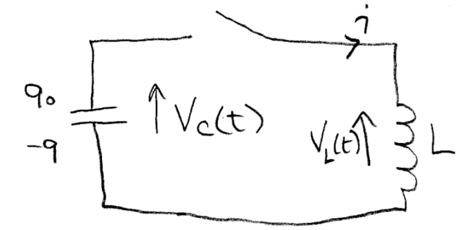
$$iAe^{i\theta} = iA\left(\cos\theta + i\sin\theta\right) = A\left(i\cos\theta - \sin\theta\right) \quad (2)$$

$$(1) = (2)$$

The solution to problem 3 can be found in Chapter 1 Example 5 in Hirose & Lonngren.

H&L 1.10

$V_c = \frac{q}{C}$ $\quad V_L = L\frac{di}{dt}$



a) From Eq. 1.35 $\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$

whose solution is 1.36 $q(t) = q_0 \cos\omega t$ where q_0 is the initial charge on the capacitor

if the initial charge on the capacitor is q_0, then the initial voltage on the capacitor is given by $V_0 = \frac{q_0}{C}$

so 1.36 becomes $q(t) = V_0 C \cos\omega t$

$i = -\frac{dq}{dt} = -\omega V_0 C(-\sin\omega t)$ where $\omega = \frac{1}{\sqrt{LC}}$

simplifying:

$$i = +\frac{1}{\sqrt{LC}}(V_0 C)\sin\omega t = \sqrt{\frac{C}{L}}V_0 \sin\omega t$$

$$= \frac{V_0}{\sqrt{L/C}}\sin\omega t.$$

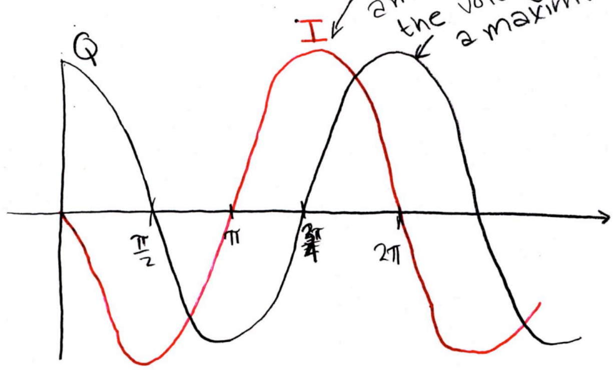
b) Inductance has units of $H = 1\,J/A^2$
Capacitance has units of $F = 1\,C/V$
Resistance has units of $\Omega = 1\,V/A$

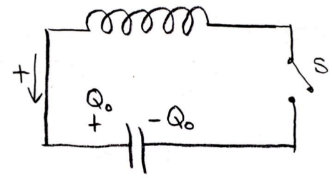
see Appendix A of Tipler & Mosca

$$\therefore \sqrt{\frac{L}{C}} = \sqrt{\frac{J/A^2}{C/V}} = \sqrt{\frac{J}{A^2}\cdot\frac{V}{C}}$$ wher $1V = J/C$ $\Rightarrow J = VC$

$$= \sqrt{\frac{V\cancel{C}}{A^2}\cdot\frac{V}{\cancel{C}}} = \sqrt{\frac{V^2}{A^2}} = \frac{V}{A} = \Omega$$

T&M 29.37





b) Eq. 29.38 $Q = Q_{peak} \cos \omega t$

$$i = \frac{dQ}{dt} = -\underbrace{\omega Q_{peak}}_{I_{peak}} \sin \omega t \qquad \text{but } \sin\theta = -\cos(\theta + \tfrac{\pi}{2})$$

$$= I_{peak} \cos(\omega t + \tfrac{\pi}{2})$$

6. $x = Ae^{-\alpha t}\cos\omega t$

$$\frac{dx}{dt} = -\alpha A e^{-\alpha t}\cos\omega t - \omega A e^{-\alpha t}\sin\omega t$$

$$\frac{d^2x}{dt^2} = +\alpha^2 A e^{-\alpha t}\cos\omega t + \alpha\omega A e^{-\alpha t}\sin\omega t$$
$$-\omega^2 A e^{-\alpha t}\cos\omega t + \omega\alpha A e^{-\alpha t}\sin\omega t$$
$$= (\alpha^2-\omega^2)Ae^{-\alpha t}\cos\omega t + 2\omega\alpha A e^{-\alpha t}\sin\omega t$$

plugging these expressions into

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

divide equation by common factors

$$(\alpha^2-\omega^2)\cancel{Ae^{-\alpha t}}\cos\omega t + 2\omega\alpha\cancel{Ae^{-\alpha t}}\sin\omega t$$
$$-\alpha\gamma\cancel{Ae^{-\alpha t}}\cos\omega t - \gamma\omega\cancel{Ae^{-\alpha t}}\sin\omega t$$
$$+\omega_0^2\cancel{Ae^{-\alpha t}}\cos\omega t = 0$$

sine and cosine coefficients must independently add to 0.

$(\alpha^2-\omega^2) - \alpha\gamma + \omega_0^2 = 0$ (cos terms)

$2\omega\alpha - \gamma\omega = 0$ (sin terms)

$$\Rightarrow \alpha = \gamma/2$$

plugging this into the equation for the cos terms

$$\frac{\gamma^2}{4} - \omega^2 - \frac{\gamma^2}{2} + \omega_0^2 = 0 \Rightarrow \omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

$$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

so the solution $Ae^{-\alpha t}\cos\omega t$ holds
where $\alpha = \gamma/2$ and $\omega = \sqrt{\omega_0 - \gamma^2/4}$

Note that we did this same problem in class assuming an exponential solution, and we got the solution $z = Ae^{-bt}e^{i(at+\phi)}$
where $b = \gamma/2$ and $a = \sqrt{\omega_0^2 - \gamma^2/4}$
Note that if we take the real part of that exponential solution, we get the same solution, to within a phase shift!!
I encourage you to try the problem again with $z = Ae^{-\gamma t}e^{i\omega t}$!!