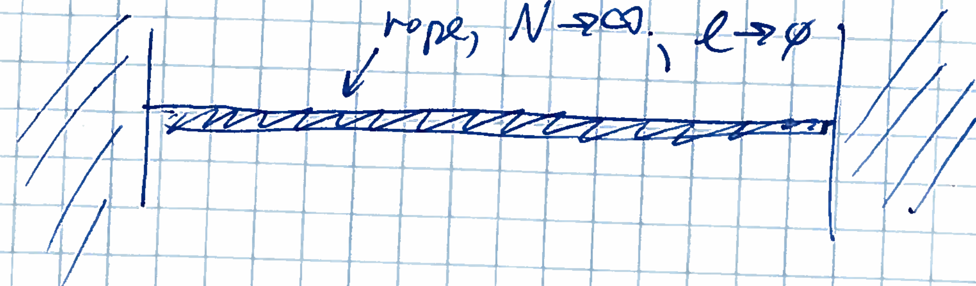
Continuous Systems — Wave Equation

We can model a continuous system, like a rope, as being a limit where the number of particles goes to infinity and ℓ goes to zero.



For N masses, our equation of motion was

$$\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = \phi$$

or

$$\ddot{y}_p = \omega_0^2 (y_{p+1} - y_p) - \omega_0^2 (y_p - y_{p-1})$$

Recall $\omega_0^2 = \frac{T}{m\ell}$

$$\therefore \quad m\ddot{y}_p = \frac{T}{\ell}(y_{p+1} - y_p) - \frac{T}{\ell}(y_p - y_{p-1})$$

Divide by ℓ:

$$\frac{m}{\ell}\ddot{y}_p = \frac{T}{\ell}\left(\frac{y_{p+1} - y_p}{\ell}\right) - \frac{T}{\ell}\left(\frac{y_p - y_{p-1}}{\ell}\right)$$

As ℓ goes to zero:

$$\lim_{\ell \to \phi}\left(\frac{y_{p+1} - y_p}{\ell}\right) \Rightarrow \lim_{\ell \to \phi}\left(\frac{y(x+\ell) - y(x)}{\ell}\right) = \left.\frac{dy}{dx}\right|_{x+\frac{\ell}{2}}$$

$$\lim_{\ell \to \phi}\left(\frac{y_p - y_{p-1}}{\ell}\right) \Rightarrow \lim_{\ell \to \phi}\left(\frac{y(x) - y(x-\ell)}{\ell}\right) = \left.\frac{dy}{dx}\right|_{x-\frac{\ell}{2}}$$

Also, let $\frac{m}{\ell} = \rho$ = mass density per unit length

Then

$$\cancel{\rho} \quad \rho \ddot{y}(x) = \frac{T}{\ell}\left[\frac{dy}{dx}\Big|_{x+\frac{\ell}{2}} - \frac{dy}{dx}\Big|_{x-\frac{\ell}{2}}\right]$$

$$\cancel{\rho} \quad \rho \frac{d^2y}{dt^2} = T \lim_{\ell \to 0} \underbrace{\left[\frac{\frac{dy}{dx}\Big|_{x+\frac{\ell}{2}} - \frac{dy}{dx}\Big|_{x-\frac{\ell}{2}}}{\ell}\right]}_{\frac{d^2y}{dx^2}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\rho}{T}\frac{d^2y}{dt^2}}$$ "Wave Equation"

This is the Eq. of Motion for a continuous system of masses. It is Newton's 2nd Law.

Solution: The normal modes we can get by allowing $N \to \infty$ in the N-mass system, while $\ell \to 0$ such that $(N+1)\ell = L$ = total length

For N particles,

$y_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right)$ Now $p\ell = x$ = distance along rope

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i.e. $y_n(x) = C_n \sin\left(\frac{\overbrace{(p\ell)}^{x}\pi n}{\underbrace{\ell(N+1)}_{L = \text{total length}}}\right) = \boxed{C_n \sin\left(\frac{n\pi x}{L}\right)}$ $n = 1, 2, \ldots \infty$

Amplitude relationship for normal modes of of continuous system

The frequencies are

$$\omega_n = 2\,\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi\ell}{2\underbrace{(N+1)\ell}_{L}}\right) = 2\omega_0 \sin\left(\frac{n\pi\ell}{2L}\right)$$

In the limit when $\ell \to \emptyset$, $\sin\left(\frac{n\pi\ell}{2L}\right) \to \frac{n\pi\ell}{2L}$

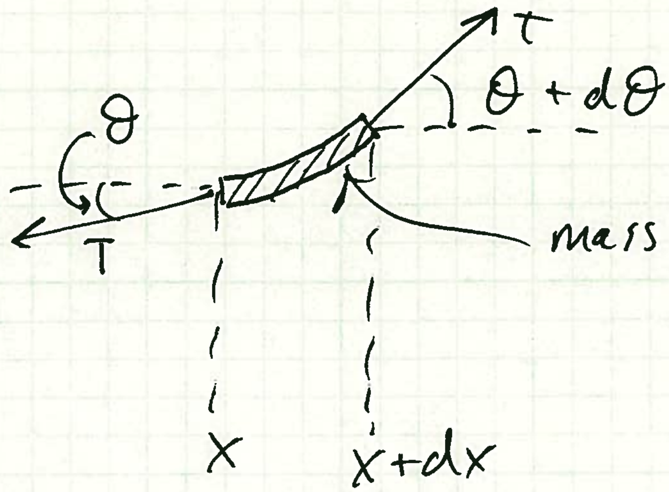
$$\omega_n = 2\,\omega_0\left(\frac{n\pi\ell}{2L}\right)$$

$$\omega_0 = \sqrt{\frac{T}{m\ell}} = \sqrt{\frac{T/\ell^2}{m\ell}} = \frac{1}{\ell}\sqrt{\frac{T}{\rho}}, \quad \rho = \frac{m}{\ell}$$

$$\boxed{\omega_n = \sqrt{\frac{T}{\rho}}\;\frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \infty}$$

Another Derivation of the Wave Equation.

Consider a short segment of rope:



mass density $\rho = \frac{\text{mass}}{\text{length}}$

The force on the segment of rope has two components:

$$F_y = T\sin(\theta + d\theta) - T\sin(\theta)$$
$$F_x = T\cos(\theta + d\theta) - T\cos(\theta)$$

For small $d\theta$, $\sin(\theta + d\theta) \approx \sin\theta + d\theta$
$\cos(\theta + d\theta) \approx \cos\theta$

So $F_y \approx T\,d\theta$
$F_x \approx \emptyset$

So the Eq. of Motion in the y direction is

$$T\,d\theta = (m)\ddot{y} = (\rho\,dx)\ddot{y}$$

Also $\tan\theta = \dfrac{\partial y}{\partial x}$ ← take derivative w/respect to θ

$$\underbrace{\frac{d}{d\theta}\left(\tan\theta\right)} = \frac{d}{d\theta}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial x^2}\frac{\partial x}{\partial \theta}$$

$$\underbrace{\sec^2\theta} = \frac{\partial^2 y}{\partial x^2}\frac{\partial x}{\partial \theta}$$

$\sec^2\theta \simeq 1$ because θ is small

$$\therefore \quad 1 = \frac{\partial^2 y}{\partial x^2}\frac{\partial x}{\partial \theta}$$

$$d\theta = \frac{\partial^2 y}{\partial x^2}\,dx$$

So we have

$$T\,d\theta = (\rho\,dx)\,\ddot{y}$$

$$T\left(\frac{\partial^2 y}{\partial x^2}\,dx\right) = (\rho\,dx)\,\ddot{y}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\,\ddot{y} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}}$$

$\leftarrow$ Equation of Motion for a continuous string

"Wave Equation.

Summarizing

Equation of Motion: $$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

Normal Mode Amplitude Relationship: $y_n(x) = C_n \sin\left(\frac{n\pi}{L}\right.$

Normal Mode Frequencies: $\omega_n = \sqrt{\frac{T}{\rho}}\,\frac{n\pi}{L}$, $n = 1, 2, 3, \ldots \infty$

General Solution:

$$y(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

C_n = complex (real & imaginary parts)
(or amplitude & phase)

Required Initial conditions: $\begin{cases} y(x, t=\emptyset) = ? \\ \dot{y}(x, t=\emptyset) = ? \end{cases}$

These determine the real & imaginary parts of C_n

How to determine the $\{c_n\}$

First write the real part of the solution:

Let $c_n \equiv a_n + i b_n$, a_n real, b_n real.

Then $$\text{Real}\left[y(x,t)\right] = \text{Real}\left[\sum_{n=1}^{\infty}(a_n + i b_n)\sin\left(\frac{n\pi x}{L}\right)e^{i\omega_n t}\right]$$

$$y(x,t) = \sum_{n=1}^{\infty}\left[a_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) - b_n \sin\left(\frac{n\pi x}{L}\right)\sin(\omega_n t)\right]$$

~~For the velocity~~

And at $t=0$,

$$\boxed{y(x,t=0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)} \quad (1)$$

For the velocity,

$$\text{Real}\left[\dot{y}(x,t)\right] = \text{Real}\left[\sum_{n=1}^{\infty}(i\omega_n)(a_n + i b_n)\sin\left(\frac{n\pi x}{L}\right)e^{i\omega_n t}\right]$$

$$\dot{y}(x,t) = \sum_{n=1}^{\infty}\left[(-\omega_n) b_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) + (-\omega_n) a_n \sin\left(\frac{n\pi x}{L}\right)\sin(\omega_n t)\right]$$

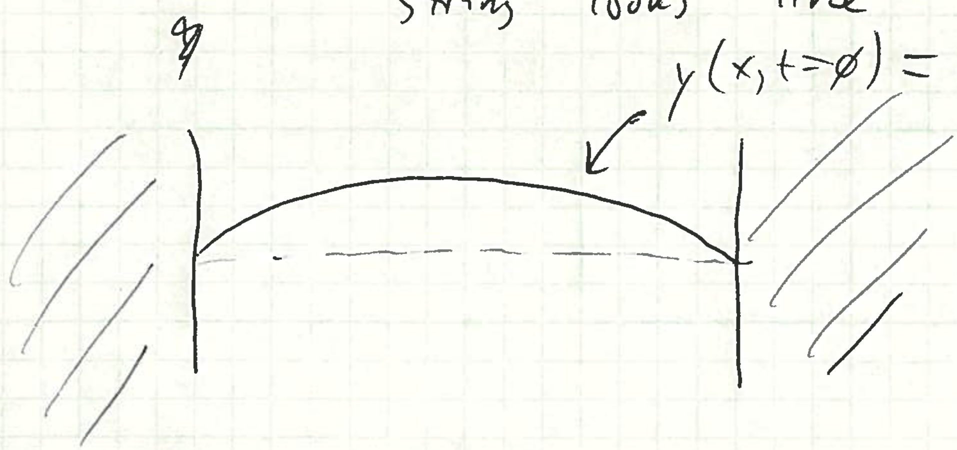
At $t=0$,

$$\boxed{\dot{y}(x,t=0) = \sum_{n=1}^{\infty} -\omega_n b_n \sin\left(\frac{n\pi x}{L}\right)} \quad (2)$$

Our job is to determine the $\{a_n\}$ and $\{b_n\}$ given the initial position $y(x, t=0)$ and velocity $\dot{y}(x, t=0)$.

A simple case: Suppose at $t=0$, the string looks like

$$y(x, t=0) = A\sin\left(\frac{\pi x}{L}\right)$$



And it's velocity is zero everywhere. What ~~are the~~ is the solution as time goes forward?

Answer: by inspection, we must have

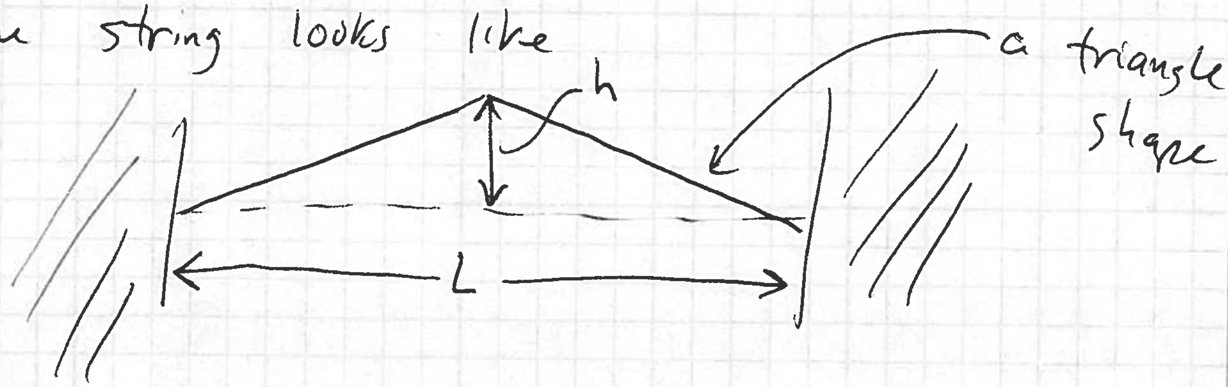
$$a_1 = A, \quad b_1 = 0$$
$$a_2 = 0 \quad b_2 = 0$$
$$a_3 = 0 \quad b_3 = 0$$
$$\vdots \qquad \vdots$$

Then the complete solution is

$$y(x,t) = A\sin\left(\frac{\pi x}{L}\right)\cos(\omega_1 t), \quad \omega_1 = \sqrt{\frac{T}{\rho}}\left(\frac{\pi}{L}\right)$$

This happens to be a perfect normal mode initial condition, so the solution is particularly simple

But suppose the initial condition is more complicated. Suppose, for example that at $t=0$ the string looks like



In other words,

$$y(x,t=0) = \begin{cases} \frac{2h}{L}x, & 0 \le x \le L/2 \\ \frac{2h}{L}(L-x), & \frac{L}{2} \le x < L \end{cases}$$

Let's also assume that it starts from rest:

$$\dot{y}(x,t=0) = 0 .$$

What are the $\{a_n\}$ and $\{b_n\}$ now? How does the string evolve in time? What is $y(x,t)$ as time goes forward?

This could be a hard problem, because there are an <u>infinite</u> <u>number</u> of $\{a_n\}$ and $\{b_n\}$.

But let's consider <u>just</u> <u>one</u> of them. Suppose we want to know the <u>seventh</u> a and <u>seventh</u> b: What is a_7?

and What is b_7?

There is a fantastic trick for determining one coefficient. Griffiths calls it "Fouriers Trick". It goes like this.

Start with Eq 1: $y(x,t=0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$

To find a_7, multiply both sides by $\boxed{\sin\left(\frac{7\pi x}{L}\right)}$:

$$\sin\left(\frac{7\pi x}{L}\right) \underbrace{y(x,t=0)}_{y(x)} = \left[\sin\left(\frac{7\pi x}{L}\right)\right] \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

Now integrate both sides over x from zero to L:

$$\int_0^L \sin\left(\frac{7\pi x}{L}\right) y(x)\, dx = \int_0^L \sin\left(\frac{7\pi x}{L}\right)\left[\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\left(\frac{7\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx}$$

Here's the beautiful part: This integral is very simple:

$$\int_0^L \sin\left(\frac{7\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & \text{for } n \neq 7 \\ \frac{L}{2}, & \text{for } n = 7 \end{cases}$$

This means that the infinite sum has only one non-zero term:

$$\sum_{n=1}^{\infty} a_n \int_0^L \sin\left(\frac{7\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = a_7 \left(\frac{L}{2}\right)$$

only the 7th term survives.

Summarizing,

$$\int_0^L \sin\left(\frac{7\pi x}{L}\right) \underbrace{y(x)}_{\substack{\uparrow \\ \text{initial} \\ \text{condition}}} dx = a_7\left(\frac{L}{2}\right)$$

or

$$\boxed{a_7 = \frac{2}{L}\int_0^L \sin\left(\frac{7\pi x}{L}\right) \overbrace{y(x)}^{\text{initial condition}} dx}$$

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This equation tells us how to calculate a_7 given the initial conditions for $y(x, t=0)$.

But a_7 is just one of an infinite number of coefficients to calculate! How can we get them all?

Answer: In general, the m^{th} coefficient (a_m) will be given by

$$\boxed{a_m = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) \underbrace{y(x)}_{\text{initial condition}} dx}$$

This is the essential feature of Fourier's Trick: it allows us to calculate all the coefficients by evaluating this integral.