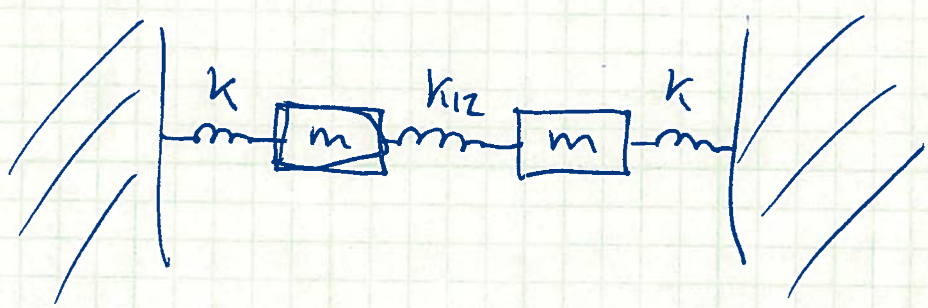
What is a normal mode?

A normal mode is a special type of motion for a multi-particle harmonic system. It's defining characteristic is that the time evolution is very simple: <u>every particle oscillates at the same frequency</u>

For a system to satisfy this condition, the amplitudes & phases of the various particles must be related to each other. So to find a normal mode, we must do 2 things:

1. determine which frequencies are normal-mode frequencies.
2. determine the relationship between the amplitudes of motion for ~~that~~ each normal mode

Example Coupled Mechanical Oscillator (2 particle system)



We solved the equation of motion and found 2 normal frequencies:

$$\omega_s = \sqrt{\frac{k}{m}} = \text{"small frequency"}$$

$$\omega_L = \sqrt{\frac{k+2k_{12}}{m}} = \text{"large frequency"}$$

We also found the amplitude relationship:

For $\underline{\omega_s}$:

$$\boxed{\begin{array}{l} x_1 = B_s e^{i\omega_s t} \\ \quad \updownarrow \text{ same amplitude and phase} \\ x_2 = B_s e^{i\omega_s t} \end{array}}$$

For $\underline{\omega_L}$:

$$\boxed{\begin{array}{l} x_1 = B_L e^{i\omega_L t} \\ \quad \updownarrow \text{ same amplitude, opposite phase} \\ x_2 = -B_L e^{i\omega_L t} \end{array}}$$

opposite phase

(Recall that B_s & B_L are complex, so the phase at $t=\emptyset$ is absorbed into B_s & B_L.)

Notice that our 2-particle system has 2 normal modes. In general, an N-particle system will have N-normal modes.

We determined B_L & B_S for a particular set of initial conditions:

$$x_1(t=0) = a \quad , \quad \dot{x}_1(t=0) = 0$$
$$x_2(t=0) = 0 \quad , \quad \dot{x}_2(t=0) = 0$$

Complete solution: (Real part)

$$x_1(t) = \frac{a}{2}\cos(\omega_s t) + \frac{a}{2}\cos(\omega_L t)$$

$$x_2(t) = \frac{a}{2}\cos(\omega_s t) - \frac{a}{2}\cos(\omega_L t)$$

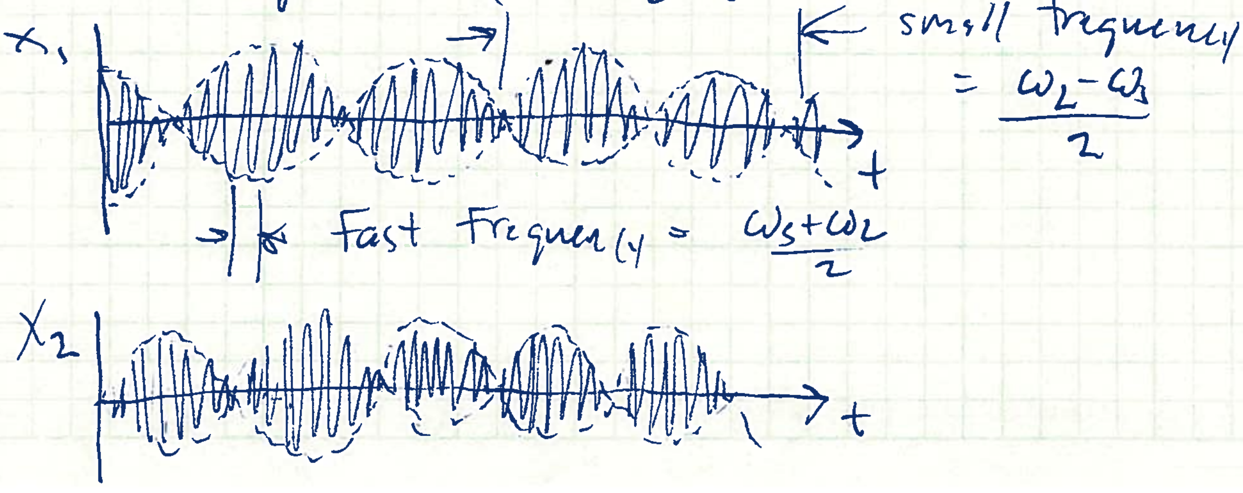
Question: What does this solution look like?

Answer: Re-write it using a (dreaded) trig identity:

$$x_1(t) = a\cos\left(\frac{(\omega_L - \omega_S)}{2}t\right)\cos\left(\left(\frac{\omega_L+\omega_S}{2}\right)t\right)$$

$$x_2(t) = a\sin\left(\frac{(\omega_L - \omega_S)}{2}t\right)\sin\left(\left(\frac{\omega_L+\omega_S}{2}\right)t\right)$$

So we have two harmonic functions multiplied together. There is a "fast oscillation" whose frequency is the average of ω_L & ω_S. But the amplitude goes up and down with a slow frequency $\left(\frac{(\omega_L - \omega_S)}{2}\right)$.



Question

Why do we care about normal modes?

Answer: Two reasons

① The general solution, valid for any initial conditions, can be written as a sum over normal modes:

$$\left.\begin{aligned} x_1 &= B_S e^{i\omega_S t} + B_L e^{i\omega_L t} \\ x_2 &= B_S e^{i\omega_S t} - B_L e^{i\omega_L t} \end{aligned}\right\} \text{sum over normal mode solutions}$$

Any valid motion of the system can be described ~~also~~ by specifying four initial conditions: Real & Imag. parts of B_S & real & imaginary parts of B_L.

② The time-evolution of each normal mode is extremely simple: simply multiply by $e^{i\omega_i t}$ for each mode.

This is easier to see if we simplify our notation. Let $\vec{x} \equiv (x_1, x_2)$ be a vector which describes the current position of m_1 & m_2.

Rename: $B_S = a_1$, $\omega_S = \omega_1$
$B_L = a_2$, $\omega_L = \omega_2$

Then

$$\boxed{\begin{aligned} x_1(t) &= a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} \\ x_2(t) &= a_1 e^{i\omega_1 t} - a_2 e^{i\omega_2 t} \end{aligned}}$$

With vector notation I can combine two equations into one:

time evolution

$$\left(x_1(t),\ x_2(t)\right) = \underbrace{a_1}_{\text{initial conditions}}\underbrace{(1,1)}_{\text{constant vector}} e^{i\omega_1 t} + \underbrace{a_2}_{\text{initial conditions}}\underbrace{(1,-1)}_{\text{constant vector}} e^{i\omega_2 t}$$

I can simplify the notation further if I define

$$\vec{q}_1 \equiv \text{constant vector} \equiv (1,1)$$

$$\vec{q}_2 \equiv \text{constant vector} \equiv (1,-1)$$

Then $\boxed{\vec{x}(t) = a_1\vec{q}_1 e^{i\omega_1 t} + a_2\vec{q}_2 e^{i\omega_2 t}}$

Should we make it even simpler? Use summation notation:

$$\boxed{\vec{x}(t) = \sum_{n=1}^{2} a_n \vec{q}_n e^{i\omega_n t}}$$

This equation says exactly the same thing as our original general solution, but it is written more compactly and elegantly.

For example, we still need 4 initial conditions to specify $Re(a_1)$, $Im(a_1)$, $Re(a_2)$, $Im(a_2)$.

The vectors $\vec{q}_1$ & $\vec{q}_2$ are the "normal mode eigenvectors". They ~~a~~ are fixed, constant vectors which describe the fixed relationship between the amplitudes of ~~x_1~~ X_1 & ~~x_2~~ X_2 in each normal mode.

$\vec{q}_1 = (1,1)$ = "X_1 & X_2 have the same amplitude and phase in this mode" (symmetric mode)

$\vec{q}_2 = (1,-1)$ = "X_1 & X_2 have the same amplitude, but a phase difference of 180°, in this mode" (antisymmetric mode)

In general, the system is ~~[illegible]~~ not in a single normal mode, but is in a sum, or superposition, of normal modes. The fixed relationship between amplitudes of X_1 & X_2 will only occur when the system happens to be in a pure normal mode.

~~The amp~~

a_1 & a_2, which are determined by the initial conditions, are called the "normal coordinates". They describe "how much of each normal mode" is in the ~~[illegible]~~ motion.

If we want to <u>simplify</u> further we could absorb the time evolution factor into a_1 & a_2:

$$\vec{x}(t) = \sum_{n=1}^{2} a_n \vec{q}_n e^{i\omega_n t} = \sum_{n=1}^{2} \underbrace{\left(a_n e^{i\omega_n t}\right)}_{a_n(t) \equiv a_n e^{i\omega_n t}} \vec{q}_n$$

$$= \sum_{n=1}^{2} a_n(t) \vec{q}_n = a_1(t)\vec{q}_1 + a_2(t)\vec{q}_2$$

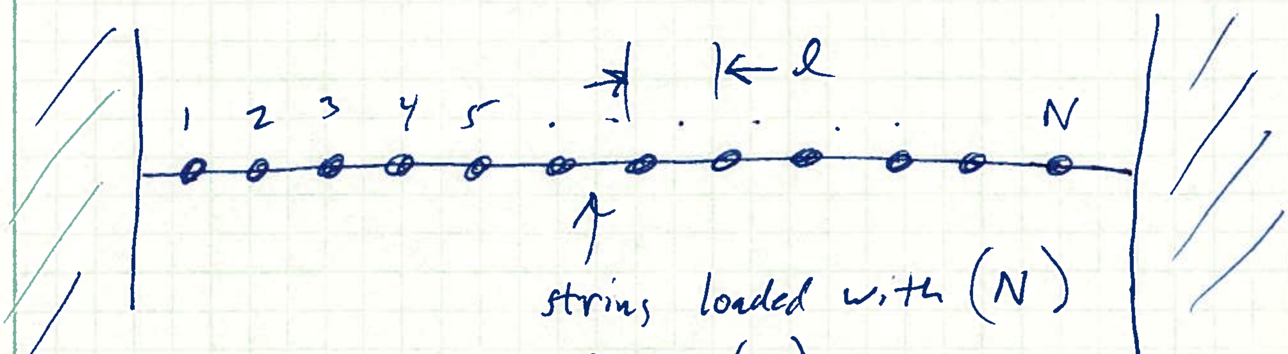
Since $a_n(t) \equiv a_n e^{i\omega_n t}$, we see that each normal mode evolves in time by picking of a <u>phase factor</u> of $e^{i\omega_n t}$. Notice that the <u>magnitude</u> of each normal mode component does not change, only its phase:

$$|a_n(t)| = |a_n e^{i\omega_n t}| = |a_n| \underbrace{|e^{i\omega_n t}|}_{\text{magnitude} = 1} = |a_n| = \text{Constant}.$$

Therefore, whatever normal modes we have at $t = \emptyset$, we will have forever. Normal modes do not appear or dissappear as time goes forward. (This is because we <u>assumed</u> <u>no drag forces</u> and <u>no driving forces</u> either. Drag forces would cause the amplitudes to decay, driving forces would cause them to grow (transient effect).)

The loaded string: N coupled oscillators and transverse motion.

For a many particle system, the normal modes are easier to visualize if the motion is transverse to the direction of the springs (instead of in the same direction.) So let's switch to transverse motion.

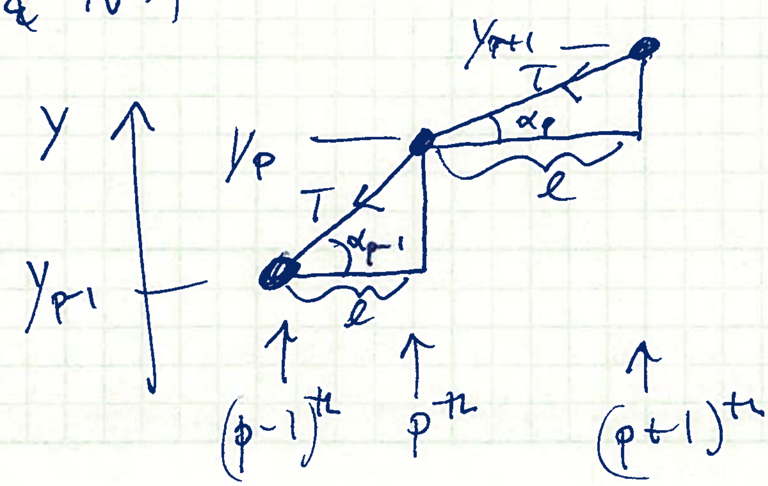


string loaded with (N) masses (m).

Evenly spaced. (distance $= l$)

String Tension $= T$

Consider ~~the pth~~ one particular ~~[illegible]~~ mass, let it be the p^{th} mass (something between 1 & N.)



$l =$ distance between masses

The neighboring masses exert a restoring force in the y direction through the string tension

$$F_y^{(p)} = -T\sin(\alpha_{p-1}) + T\sin(\alpha_{p+1})$$

~~For small α~~ ($F_x^{(p)}$ will equal zero, otherwise the string will move left or right.

For small displacements in the y direction, we can approximate the sine function:

$$\sin(\alpha_{p-1}) \approx \frac{y_p - y_{p-1}}{\ell}$$ (because $\sin\theta \approx \tan\theta$ for small θ

$$\sin(\alpha_p) \approx \frac{y_{p+1} - y_p}{\ell}$$

So $F_y^p \approx -\frac{T}{\ell}(y_p - y_{p-1}) + \frac{T}{\ell}(y_{p+1} - y_p)$

By Newton's 2nd Law, $F_y^p = m\ddot{y}_p$

$$\therefore \quad \ddot{y}_p + \frac{T}{m\ell}(2y_p) - \frac{T}{m\ell}y_{p-1} - \frac{T}{m\ell}y_{p+1} = 0$$

Define $\omega_0^2 \equiv \frac{T}{m\ell}$

$$\boxed{\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0}$$

Equation of motion for the p^{th} mass.

Depends on y_{p+1} & y_p ⟸ Coupled

We will look for normal mode solutions:

$$y_p = A_p e^{i\omega t}$$

normal mode: all masses go at the same frequency.

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Our job is to determine:

1. What frequencies ω is this a valid solution? (Normal frequencies.)
2. For each normal frequency, what are the relationships between the amplitudes A_p?

Substitute the guess into the equation of motion:

$$-\omega^2 A_p e^{i\omega t} + 2\omega_0^2 A_p e^{i\omega t} - \omega_0^2\left(A_{p+1} e^{i\omega t} + A_{p-1} e^{i\omega t}\right) = 0$$

or $(-\omega^2 + 2\omega_0^2) A_p - \omega_0^2 (A_{p-1} + A_{p+1}) = 0$

$$\frac{A_{p-1} + A_{p+1}}{A_p} = \underbrace{\frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}}$$

constant, independent of p (same constant for all masses.)

Make the following guess:

$$A_p = C\sin(p\theta)$$, where θ is some constant that we must determine.

Does this guess work? Try it:

$$\frac{A_{p-1} + A_{p+1}}{A_p} \stackrel{?}{=} \text{constant, independent of } p$$

insert guess

$$\frac{C\sin((p-1)\theta) + C\sin((p+1)\theta)}{C\sin(p\theta)} \stackrel{?}{=} \text{constant}$$

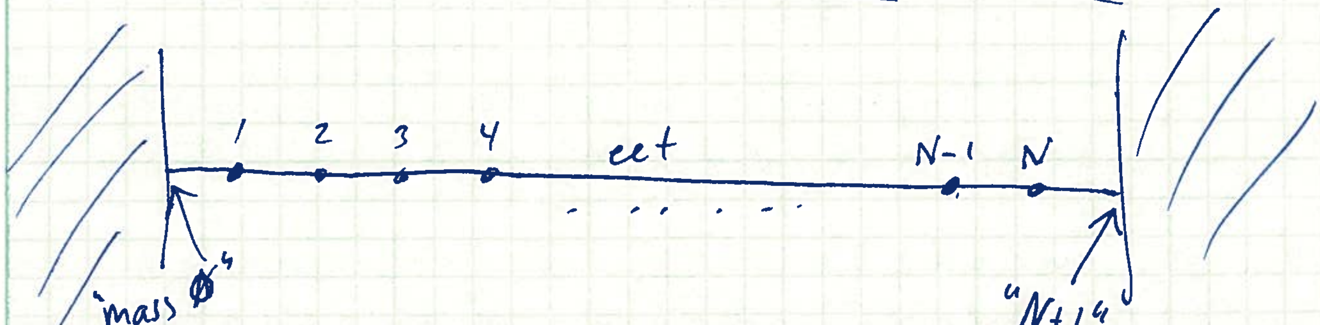
trig identity

$$\frac{2C\sin(p\theta)\cos(\theta)}{C\sin(p\theta)} \stackrel{?}{=} \text{constant, independent of } p$$

$$\boxed{2\cos(\theta) = \text{constant, independent of } p}$$ ✓ Yes

So our guess is viable. But we must determine θ and ω^2.

To fix θ, use the <u>boundary</u> <u>condition</u>:



Because the ends of the string are fixed, the amplitude ~~sh~~ A_p should go to zero for $p = \emptyset$ & $p = N+1$. Let's see if that can be made to work

$$A_p = C \sin(p\theta)$$

$$A_\emptyset = C \sin(\emptyset \theta) = \emptyset \quad \checkmark \qquad \text{yes}$$

And

$$A_{N+1} = C \sin\big((N+1)\theta\big) = \emptyset$$

$$(N+1)\theta = n\pi \quad , \quad n = 1, 2, 3, 4, \ldots$$

$$\boxed{\theta = \frac{n\pi}{N+1} \;, n = 1, 2, 3}$$

Therefore our solution for A_p is

$$\boxed{A_p = C \sin\left(\frac{pn\pi}{N+1}\right)}$$

What about the normal frequencies?

$$\underbrace{\frac{A_{p-1} + A_{p+1}}{A_p}} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$2\cos(\theta) = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$\uparrow$$

$$\frac{n\pi}{N+1}$$

$$2\cos\left(\frac{n\pi}{N+1}\right) = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$\omega^2 = 2\omega_0^2\left(1-\cos\left(\frac{n\pi}{N+1}\right)\right)$$

trig identity

$$\omega^2 = 4\omega_0^2 \sin^2\left(\frac{n\pi}{2(N+1)}\right)$$

Add an (n) subscript, because RHS depends on (n)

$$\boxed{\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)}$$

Normal Frequencies.

So we have found the normal mode solutions. The amplitude relationship is

$$\boxed{\begin{aligned} A_{pn} &= C\sin\left(\frac{pn\pi}{N+1}\right) \\ \omega_n &= 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)\end{aligned}}$$

and the normal frequencies are:

Normal Modes for a string loaded with N masses.

In this expression:

- p is an integer which tells us which mass we are talking about
- N is the number of masses ($p = 1, 2, \ldots, N$)
- n tells us which ~~of~~ ~~[illegible]~~ normal mode we are considering.
- $\omega_0 = T/ml$

Properties of the Normal Modes of the loaded string

Recall that the displacement of the p^{th} mass for a particular normal mode is

$$y_p = A_{pn} e^{i\omega t} = C \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega t}$$

~~In~~ Here we have assumed that the phase at $t=\emptyset$ is zero. If we want to allow a non-zero phase we could write

$$y_p = A_{pn} e^{i(\omega t+\delta)} \quad \text{or} \quad \cancel{y_p = \left(A_{pn} e^{i\delta}\right) e^{i(\omega t+\delta)}}$$

$$\text{or} \quad y_p = \underbrace{\left(A_{pn} e^{i\delta}\right)}_{B_{pn}} e^{i\omega t}.$$

where B_{pn} is complex.

Also, the allowed frequencies are

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

Question: How many normal modes are there?

Answer: For a system of N masses, there are N normal modes.

We can see this as follows:

$$\omega_{N+2} = 2\omega_0 \sin\left(\frac{(N+2)\pi}{2(N+1)}\right) = 2\omega_0 \sin\left[\frac{(2(N+1)-N)\pi}{2(N+1)}\right]$$

$$= 2\omega_0 \sin\left[\pi - \frac{N\pi}{2(N+1)}\right]$$

trig identity

$$= 2\omega_0 \sin\left(\frac{N\pi}{2(N+1)}\right)$$

$$= \omega_N$$

ω_{N+2} just duplicates ω_N ... it is not an independent solution.

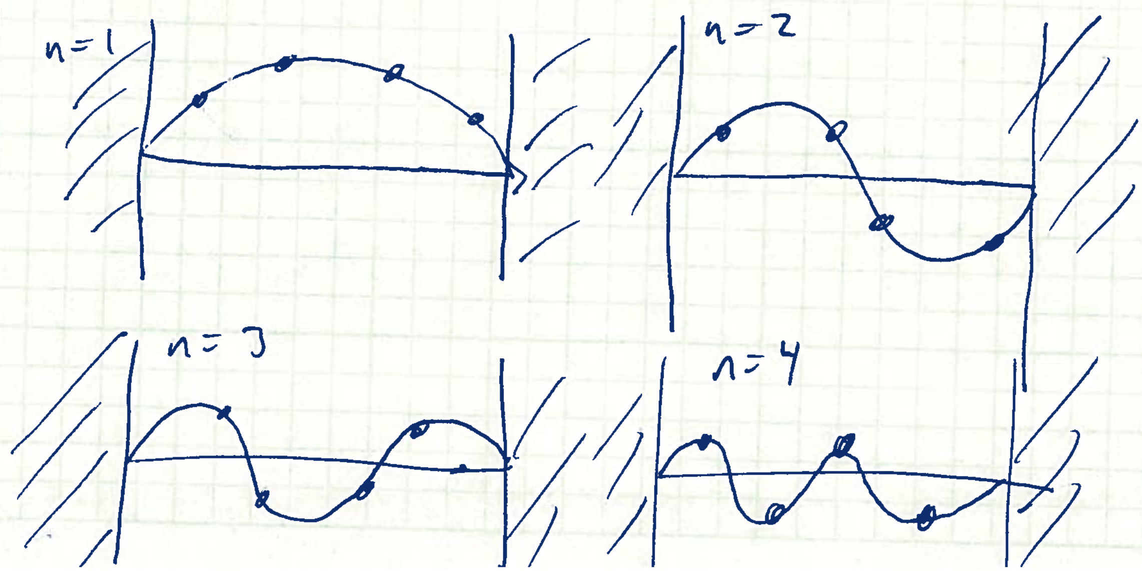
Similiarly, $\omega_{N+3} = \omega_{N-1}$

It can also be shown that the amplitude relationship (A_p) repeats itself when $n > N$.

Conclusion: There are N independent normal modes of a system of N masses on a string.

What do the modes look like?

For $N = 4$:



The general solution is a superposition of normal modes:

amplitude factor.

$$y_p = \sum_{n=1}^{N} a_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}$$

time evolution

complex coefficient, 2 free parameters each

The number of free parameters is $2N$: real & imaginary parts of a_n, where $n = 1, 2, \ldots N$. These $2N$ free parameter will be fixed by the $2N$ initial conditions: the position & velocity of every particle at $t = \emptyset$.

We can also switch to vector notation (if we like).

Let

$$\vec{y} = (y_1, y_2, y_3, \ldots, y_N)$$
$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \sin\left(\frac{3n\pi}{N+1}\right), \ldots, \sin\frac{Nn\pi}{N+1}\right)$$

Then $$\vec{y} = \sum_{n=1}^{N} a_n \vec{q}_n e^{i\omega_n t}$$

coefficient of the nth mode

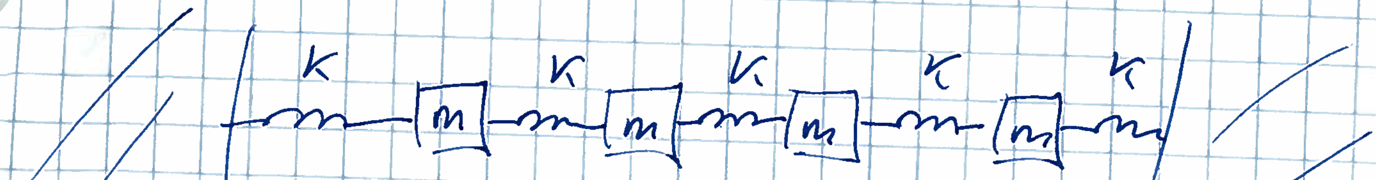
time evolution

normal mode vector

(amplitude relationship)

Longtitudinal oscillations

Consider again masses connected by springs:



Simplify by assuming a springs are identical (k).

Eq. of Motion for the p^{th} particle:

$$m\ddot{x}_p = \underset{\uparrow \atop m\omega_0^2}{k}(x_{p+1} - x_p) - \underset{\uparrow \atop m\omega_0^2}{k}(x_p - x_{p-1})$$

$$\boxed{\ddot{x}_p + 2\omega_0^2 x_p - \omega_0^2 (x_{p+1} + x_{p-1}) = \emptyset}$$

This is identical in form to the Eq. of Motion for transverse oscillations:

$$\ddot{y}_p + 2\omega_0^2 y - \omega_0^2 (y_{p+1} + y_{p-1}) = \emptyset$$

So the solution must be identical:

Normal Modes:

$$x_{pn}(t) = C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}, \quad \omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

General Solution: $$x_p = \sum_{n=1}^{N} a_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}$$