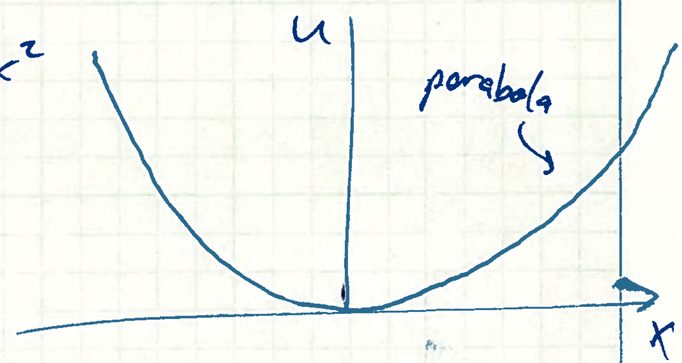
Phys 273 | Lecture 14

Exam 1 Review

Simple Harmonic Oscillator

(no damping)
(no driving force)

$F = -kx \Rightarrow U = \frac{1}{2}kx^2$



Eq. of motion:

$$\ddot{x} + \frac{k}{m}x = \emptyset$$

Define $\omega_0^2 = k/m$

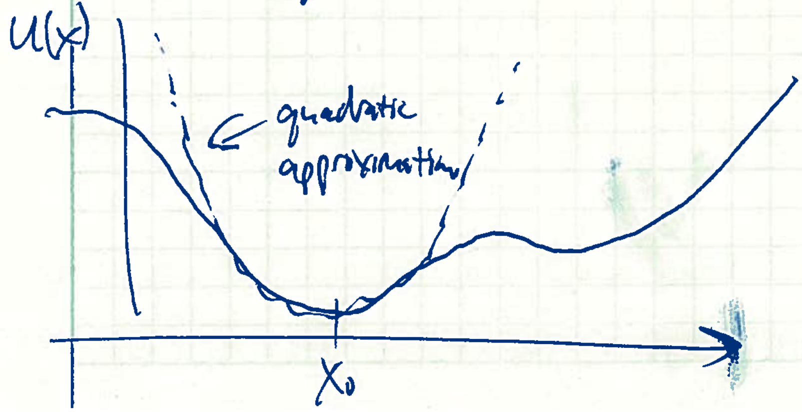
Solution: $x = Ae^{i(\omega_0 t + \delta)}$, A, δ determined by initial conditions

ω_0: natural frequency.

Small oscillations: Near a Stable equilibrium, almost any potential function is approximately quadratic.

$$U(x_0 + a) \approx U(x_0) + U'(x_0)a + \frac{1}{2}U''(x_0)a^2 + \ldots$$

x_0: equilibrium; a: displacement



$U'(x_0) = \emptyset$ since x_0 is an equilibrium point.

Frequency of small oscillations: $\omega_0 \approx \sqrt{\dfrac{U''(x_0)}{m}}$

near an equilibrium.

Plane Pendulum: $U = mgl(1-\cos\theta)$

$$\Rightarrow \omega_0 = \sqrt{g/l}$$

Complex Numbers

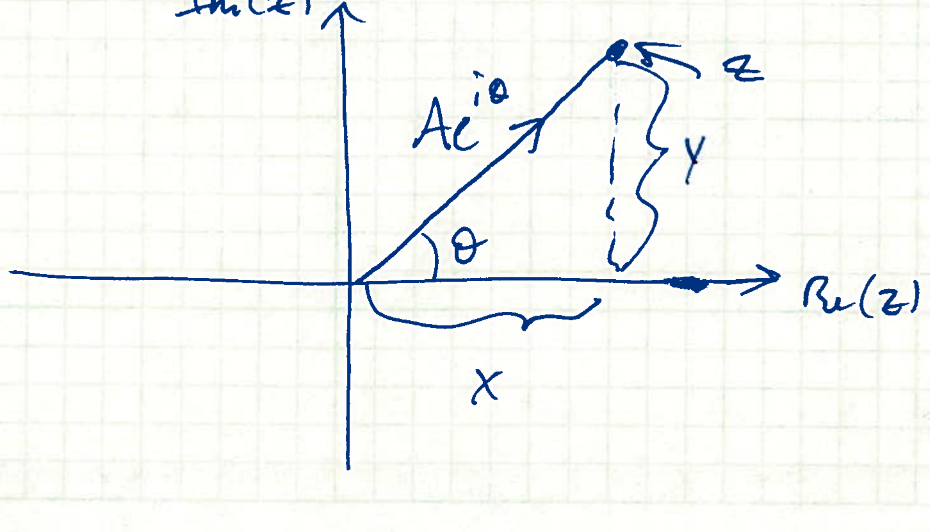
$$z = x+iy = Ae^{i\theta},$$

$$A = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x)$$

$$\text{\&} \quad x = A\cos\theta, \quad y = A\sin\theta$$

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$ Euler Formula

$$x = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad y = \frac{e^{i\theta}-e^{-i\theta}}{2}$$



$z^* \equiv$ complex conjugate $= x - iy$ (if $z = x+iy$)

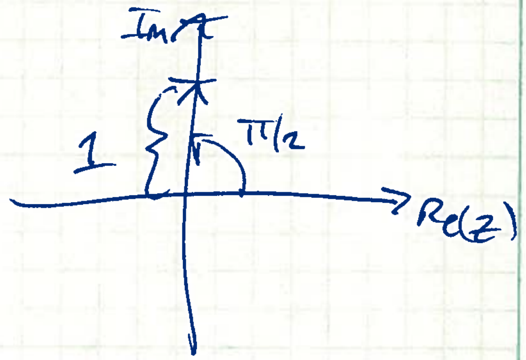
$= Ae^{-i\theta}$ (if $z = Ae^{i\theta}$)

Division: $\dfrac{1}{x+iy} = \dfrac{(x-iy)}{(x+iy)(x-iy)} = \dfrac{x-iy}{x^2+y^2}$

Also:

$zz^* = (x+iy)(x-iy) = x^2+y^2 = A^2$.

Also: $e^{i\pi/2} = i$



Also: $e^{i\pi} = -1$

Also: $e^{i\pi}+1 = \emptyset$ ← nice algebraic equation.

Forced Oscillator

Eq of Motion

$\ddot{x} + \omega_0^2 x = \dfrac{F(t)}{m}$ If $F(t) = F_0 e^{i\omega t}$

$\ddot{x} + \omega_0^2 x = \dfrac{F_0}{m} e^{i\omega t}$

"natural frequency" "driving frequency"

Also: Add damping: $F_{drag} = -bv = -b\dot{x}$

some constant

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}}$$

where $\gamma \equiv \frac{b}{m}$

$\omega_0^2 = k/m$

~~Solution:~~

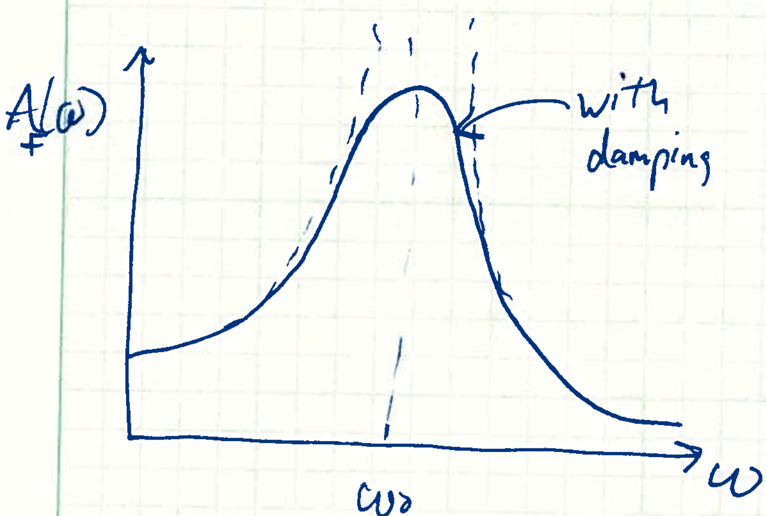
Steady-State (Long-term) Solution:

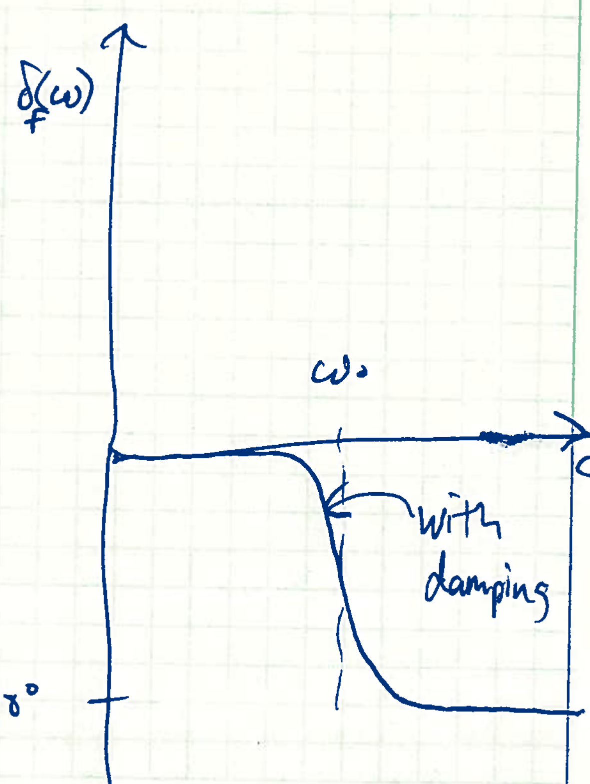
$$x(t) = A_f(\omega_f)\, e^{i(\omega_f t + \delta_f(\omega_f))}$$, no free parameters

where $$A_f(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2}}$$

and $$\delta_f(\omega) = -\tan^{-1}\left[\frac{\omega\gamma}{(\omega_0^2-\omega^2)}\right]$$

ω_f = driving frequency





(Short-term)

Including the transient Solution (Short-term Behavior):

$$x(t) = A(\omega_f)\, e^{i(\omega_f t + \delta_f(\omega_f))} + B e^{-\gamma t/2} e^{i(\omega_d t + \delta_d)}$$

$$\omega_d = \text{damped frequency} = \sqrt{\omega_0^2 - \gamma^2/4}$$

B & δ_d = Free parameters, determined by initial conditions.

Special Case: Damped oscillator, no forcing function

Then $F_0 = \emptyset$, $A(\omega_f) = \emptyset$,

$$\text{and} \quad x(t) = B \underbrace{e^{-\gamma t/2}}_{\text{damping factor}} e^{i(\omega_d t + \delta_d)}$$

Energy

- Mechanical Oscillators: $KE = \frac{1}{2} m \dot{x}^2$

  $U = \frac{1}{2} k x^2$

- Electrical Oscillators: $U_E = \int_{\text{all space}} \frac{1}{2}\varepsilon_0 |\vec{E}|\, dV$

  for capacitors $\rightarrow = \frac{1}{2} QV = \frac{1}{2} CV^2 = \frac{Q^2}{2C}$

  $U_B = \int_{\text{all space}} \frac{1}{2\mu_0} |\vec{B}|^2\, dV$

  for inductors $\longrightarrow = \frac{1}{2} L I^2$

Energy loss $\quad Q \equiv \frac{\omega_0}{\gamma}$ = unitless & very large for lightly damped oscillators.

For lightly damped oscillators

$$Q = \frac{1}{\text{Fraction of energy lost in time } t=\frac{1}{\omega_0}} = \frac{2\pi}{\text{Fraction of energy loss in one period}}$$

Energy loss: $E(t) = E_0 e^{-\gamma t} = KE(t) + U(t)$ for mechanical oscillators

$= U_E(t) + U_B(t)$ for electrical oscillators

AC circuits

Voltage rules: $|V_C| = |\frac{1}{C} Q|$ Capacitor

$|V_L| = |L \frac{dI}{dt}|$ Inductor

$|V_R| = |IR|$ Resistor

Simple LC circuit: $\omega_0 = \frac{1}{\sqrt{LC}}$ (simple harmonic oscillator)

Impedances: $Z_R = R$

$Z_L = i\omega L$

$Z_C = \frac{-i}{\omega C}$

Series Combination: $Z_{series} = Z_1 + Z_2$

Parallel Combination: $Z_{parallel} = \left[(Z_1)^{-1} + (Z_2)^{-1}\right]^{-1}$

For any element or combination of elements,

$$\vec{V} = \vec{I} Z \leftarrow \text{impedance, possibly complex.}$$

Voltage phasor

current phase

~~If Z~~ In general Z is complex, which means that there is a phase offset between $\vec{V}$ & $\vec{I}$. The ~~ratio~~ ratio of $\frac{V_0}{I_0} = |Z|$.

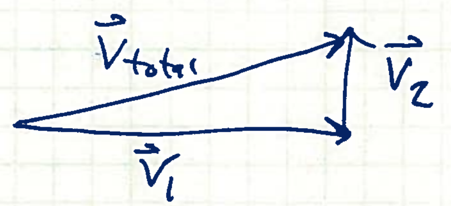
$$\vec{V}_L = \vec{I}_L \underbrace{(i\omega L)}$$

← this means V_L leads I_L by 90°.

$$\vec{V}_C = \vec{I}_C \underbrace{\left(\frac{-i}{\omega C}\right)}$$

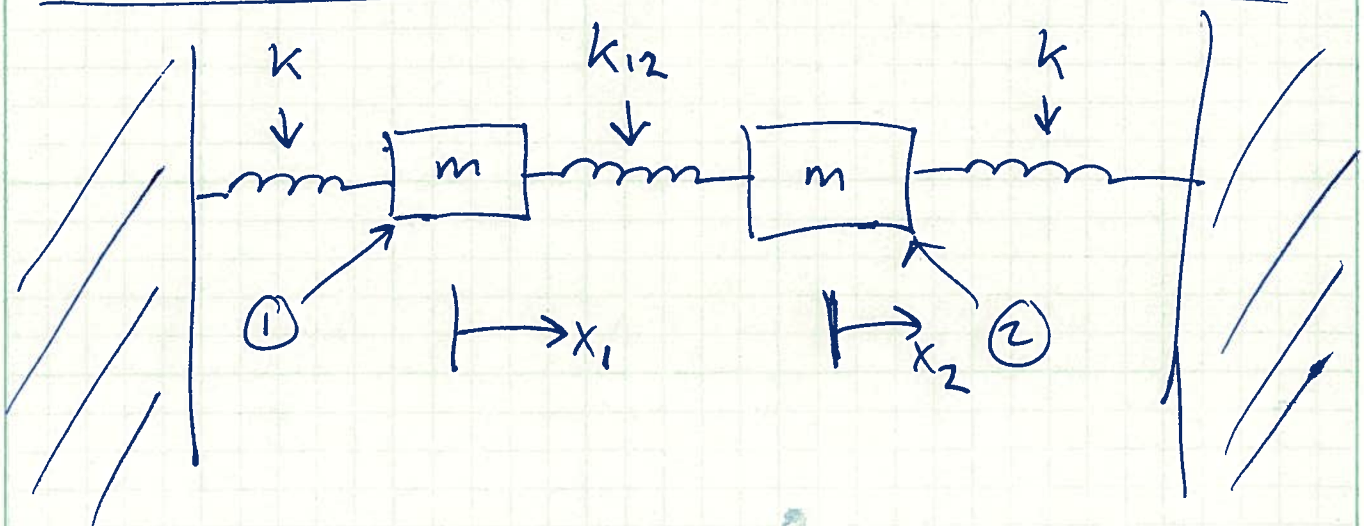
← this means V_C lags I_C by 90°

Phasor Diagrams show the geometric relationships between $\vec{V}$ & $\vec{I}$ for a circuit. This can be very useful when combined with a sum rule like $\vec{V}_{total} = \vec{V}_1 + \vec{V}_2$ or $\vec{I}_{total} = \vec{I}_1 + \vec{I}_2$



$\therefore |V_{total}| = \sqrt{V_1^2 + V_2^2}$

Two coupled Mechanical Oscillators



3 springs, 2 spring constants.

k: 2 springs which connect masses to the walls.

k_{12}: the one spring between the two masses.

Position variables: x_1 = displacement of 1 from equilibrium.

x_2 = displacement of 2 from equilibrium.

Force on 1: $F_1 = -kx_1 - k_{12}(x_1 - x_2) = m\ddot{x}_1$

" 2: $F_2 = -kx_2 + k_{12}(x_1 - x_2) = m\ddot{x}_2$

Newton's 2nd Law

Equations of Motion

$$\boxed{\begin{aligned} m\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 &= 0 \\ m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 &= 0 \end{aligned}}$$

Coupled Differential Equations.

note that x_1 & x_2 appear in both equations.
We must solve two equations simultaneously.

For the simple harmonic oscillator, we had ~~one degree of freedom~~ / one position variable, and we found one natural frequency.

Here we have two position varibles. We will still try to find a harmonic solution where both x_1 and x_2 oscillate at the same frequency. We will find 2 such frequencies.

Guessed Solution:

$$x_1 = A_1 e^{i(\omega t+\delta_1)}$$

$$x_2 = A_2 e^{i(\omega t+\delta_2)}$$

The key feature of this guess is that we are assuming that both x_1 and x_2 have the same frequency.

Normal Mode: In a multi-particle oscillating system, if all the particles oscillate at the same frequency, we call the motion a normal mode. ~~This is analogous to the~~

Normal Frequency: The frequency of a normal mode.

To simplify notation, re-write the guessed solution:

$$x_1 = A_1 e^{i(\omega t+\delta_1)} = \underbrace{\left(A_1 e^{i\delta_1}\right)}_{\equiv B_1} e^{i\omega t} = B_1 e^{i\omega t}$$

B_1 is complex, it has two free parameters (real & imaginary parts).

Also: $x_2 = B_2 e^{i\omega t}$

complex, 2 free parameters.

Substitute the guess into the Eq. of Motion:

$$\boxed{\begin{aligned} m(-\omega^2 B_1) + (k+k_{12})B_1 - k_{12}B_2 &= \emptyset \\ m(-\omega^2 B_2) + (k+k_{12})B_2 - k_{12}B_1 &= \emptyset \end{aligned}}$$

$e^{i\omega t}$ has cancelled everywhere.

Gather terms:

$$\boxed{\begin{aligned} (k+k_{12}-m\omega^2)B_1 - k_{12}B_2 &= \emptyset \qquad \text{Eq. 1} \\ -k_{12}B_1 + (k+k_{12}-m\omega^2)B_2 &= \emptyset \qquad \text{Eq. 2} \end{aligned}}$$

To have a solution, the determinant must equal zero:

$$(k+k_{12}-m\omega^2)^2 - k_{12}^2 = \emptyset$$

2 possibilities.

or $$k + k_{12} - m\omega^2 = \pm k_{12}$$

or $$\omega = \sqrt{\frac{k+k_{12} \pm k_{12}}{m}}$$

Two normal frequencies have been found:

$$\boxed{\begin{aligned} \omega_{small} = \omega_s &= \sqrt{\frac{k}{m}} \quad \leftarrow \text{smaller frequency} \\ \omega_{large} = \omega_L &= \sqrt{\frac{k+2k_{12}}{m}} \quad \leftarrow \text{larger frequency.} \end{aligned}}$$

Small Frequency Solution:

$x_1 = B_1 e^{i\omega_s t}$, B_1 has 2 parts (real & imag.)
$x_2 = B_2 e^{i\omega_s t}$, B_2 has 2 parts (real & imag.).

~~Actually~~: 4 free parameters?

Actually, we only have 2 free parameters, because B_1 & B_2 are related in the small frequency normal mode:

Substitute $\omega_s = \sqrt{k/m}$ into Eq. 1 & Eq. 2 :

$$\boxed{\begin{aligned} (k + k_{12} - k) B_1 - k_{12} B_2 &= 0 \\ -k_{12} B_1 + k_{12} B_2 &= 0 \end{aligned}}$$

$$\left.\boxed{\begin{aligned} k_{12}(B_1 - B_2) &= 0 \\ -k_{12}(B_1 - B_2) &= 0 \end{aligned}}\right\} \Rightarrow B_1 = B_2$$ for the small frequency solution.

Call it $B_s \equiv B_1 = B_2$. Then the small frequency solution is

$$\underbrace{\boxed{\begin{aligned} x_1 &= B_s e^{i\omega_s t} \\ x_2 &= B_s e^{i\omega_s t} \end{aligned}}}$$

B_s = 2 free parameters (real & imag. parts)

"Symmetric Mode" = "Small Frequency Mode"
= both oscillators have same amplitude,

The same procedure for $\omega_L = \sqrt{\dfrac{k+2k_{12}}{m}}$

leads to

$$\boxed{B_1 = -B_2}$$

Call it $B_L = B_1 = -B_2$.

(-) sign means x_1 & x_2 are out-of-phase by 180°.

Then the large frequency solution is

$$\underbrace{\boxed{\begin{aligned} x_1 &= B_L e^{i\omega_L t} \\ x_2 &= -B_L e^{i\omega_L t} \end{aligned}}}$$

B_L has 2 free parameters (real & imag. parts).

"Anti-Symmetric Mode" = "Large Frequency Mode"
= both oscillators have same amplitude and frequency, but are out-of-phase by 180°.

Demo: G2-21 Coupled Pendula.
Show symmetric and anti-symmetric oscillations, pointing out the small and large frequency.

General Solution

The two normal modes are the simplest type of motion for the coupled oscillators. But since the Eqs. of Motion are linear, we can add together the two solutions to find a general solution which can describe any motion.

$$\boxed{\begin{aligned} x_1 &= B_S e^{i\omega_S t} + B_L e^{i\omega_L t} \\ x_2 &= B_S e^{i\omega_S t} - B_L e^{i\omega_L t} \end{aligned}}$$

The most general solution.

4 free parameters: B_S real & imag. parts
B_L real & imag. parts.

We need 4 initial conditions to specify the 4 free parameters: position & velocity of 1 at $t=0$
position & velocity of 2 at $t=0$

Let's take the real part of the solution.
Let $B_S \equiv b_S e^{i\delta_S}$ and $B_L \equiv b_L e^{i\delta_L}$. Then

$$\boxed{\begin{aligned} x_1 &= b_S \cos(\omega_S t + \delta_S) + b_L \cos(\omega_L t + \delta_L) \\ x_2 &= b_S \cos(\omega_S t + \delta_S) - b_L \cos(\omega_L t + \delta_L) \end{aligned}}$$

real solution.

$b_S, \delta_S, b_L, \delta_L$ are free parameters.

Special Case: Suppose

$x_1(t=0) = a$, $\dot{x}_1(t=0) = 0$

$x_2(t=0) = 0$, $\dot{x}_2(t=0) = 0$.

Then the $\dot{x}_1 = \emptyset$ and $\dot{x}_2 = \emptyset$ requirments are:

$$\boxed{\begin{aligned} \dot{x}_1 &= -\omega_s b_s \sin(\delta_s) - \omega_L b_L \sin(\delta_L) = \emptyset \\ \dot{x}_2 &= -\omega_s b_s \sin(\delta_s) + \omega_L b_L \sin(\delta_L) = \emptyset \end{aligned}}$$

Add these equations: $\sin(\delta_s) = \emptyset \Rightarrow \boxed{\delta_s = \emptyset}$

Substract these equations: $\sin(\delta_L) = \emptyset \Rightarrow \boxed{\delta_L = \emptyset}$

And the $x_1 = a$ & $x_2 = \emptyset$ requirements are

$$x_1 = b_s \underbrace{\cos(\overset{\delta_s = \emptyset}{\delta_s})}_{1} + b_L \underbrace{\cos(\overset{\delta_L = \emptyset}{\delta_L})}_{1} = a$$

$$x_2 = b_s \overbrace{\cos(\delta_s)} - b_L \overbrace{\cos(\delta_L)} = \emptyset$$

or $\boxed{\begin{aligned} b_s + b_L &= a \\ b_s - b_L &= \emptyset \end{aligned}} \Rightarrow \boxed{\begin{aligned} b_s &= \frac{a}{2} \\ b_L &= \frac{a}{2} \end{aligned}}$

Then the special case solution is

$$\boxed{\begin{aligned} x_1 &= \frac{a}{2}\cos(\omega_s t) + \frac{a}{2}\cos(\omega_L t) \\ x_2 &= \frac{a}{2}\cos(\omega_s t) - \frac{a}{2}\cos(\omega_L t) \end{aligned}}$$ Interfering frequencies