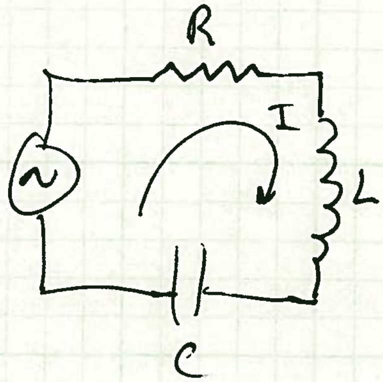
Phys 273 | Lecture 12

Driven RLC (series) circuit

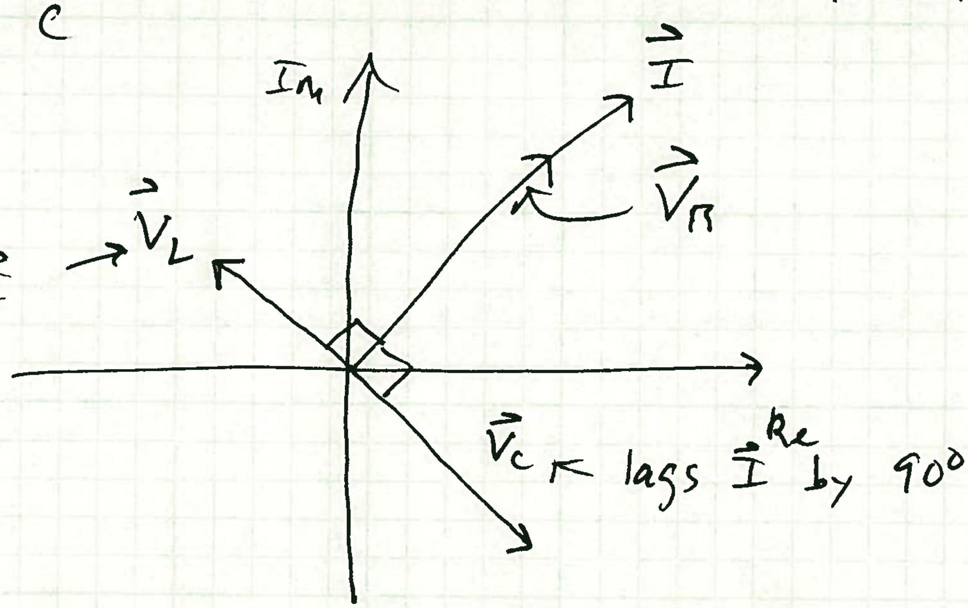
$V_S = V_0 e^{i\omega t}$



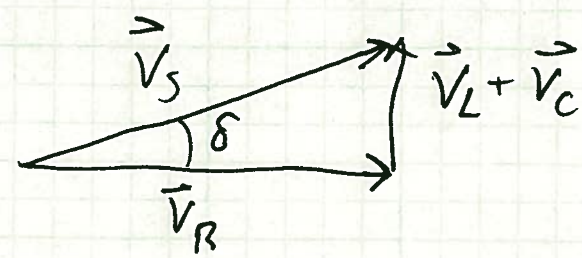
Only one current phasor: $\vec{I}$ (this circuit has only one current).

Voltage phasors can be drawn relative to $\vec{I}$: $\vec{V}_R, \vec{V}_L, \vec{V}_C$

leads $\vec{I}$ by 90°.



Voltage Loop Rule: $\vec{V}_S = \vec{V}_R + \vec{V}_L + \vec{V}_C$



Phase shift:

$$\delta = \tan^{-1}\left[\frac{|\vec{V}_L + \vec{V}_C|}{|\vec{V}_R|}\right] = \cancel{\tan^{-1}\left[\frac{\left|i\omega L I_0 - \frac{i I_0}{\omega C}\right|}{|R I_0|}\right]}$$

$$= \tan^{-1}\left[\frac{\left|i\omega L I_0 - \frac{iI_0}{\omega C}\right|}{|RI_0|}\right]$$

$$\left|i\omega L I_0 - \frac{iI_0}{\omega C}\right| = I_0\left|i\omega L - \frac{i}{\omega C}\right| = I_0\left(\omega L - \frac{1}{\omega C}\right)$$

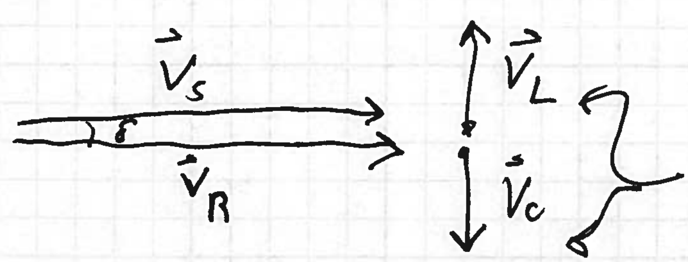
$$|RI_0| = RI_0$$

$\cancel{= \tan}$

$$= \tan^{-1}\left[\frac{I_0\left(\omega L - \frac{1}{\omega C}\right)}{I_0\,(R)}\right]$$, or, using $\omega_0^2 = \frac{1}{LC}$ and $\gamma = \frac{R}{L}$,

$$\boxed{\delta = \tan^{-1}\left[\frac{\omega^2 - \omega_0^2}{\omega\gamma}\right]}$$

What does this mean? Well, if we choose to drive the circuit with frequency $\omega = \omega_0$, then the phase difference between $\vec{V}_S$ and V_R is zero. Then the phasor diagram looks like:



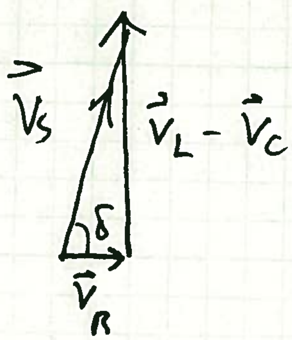
$\vec{V}_L$ & $\vec{V}_C$ are equal and opposite, when $\omega = \omega_0$.

Conversely, suppose we choose ω = very, very large.

Then $\vec{V}_C = \left(\frac{-i}{\omega C}\right)\vec{I} \approx \phi$

and $\vec{V}_L = (i\omega L)\vec{I} \approx$ large, and the

phasor diagram is

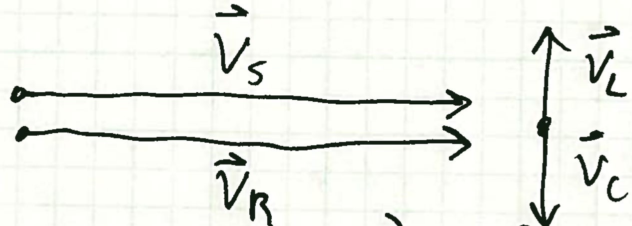


, and $\delta \to 90°$.

~~When does this~~

Question: At what driving frequency is the current maximal?

Answer: $\vec{I}$ is maximal when $\vec{V}_R$ is maximal, since $\vec{V}_R = \vec{I}R$. But $|\vec{V}_R|$ can never be larger that $|\vec{V}_S|$; ~~since they have to add to zero in the~~



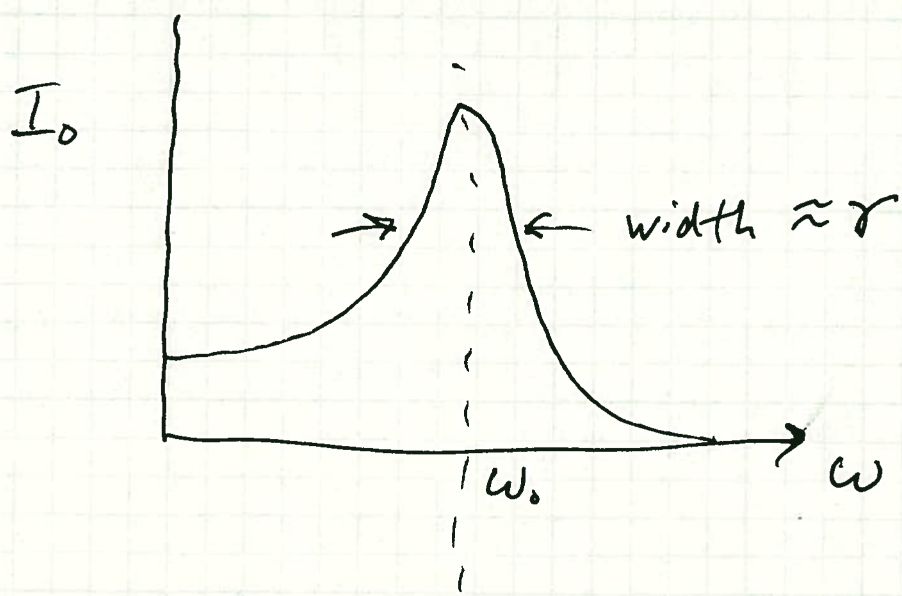
This happens when $\vec{V}_L + \vec{V}_C = \phi$

or $(i\omega L)I_0 + \left(\frac{-i}{\omega C}\right)I_0 = \phi$

or $\omega^2 = \frac{1}{LC}$

$$\boxed{\omega = \frac{1}{\sqrt{LC}} = \omega_0} \leftarrow$$ condition for maximal current.

The amplitude of the current displays a resonanance near $\omega = \omega_0$:



Series and Parallel impedance

We have the following rules for impedances in AC circuits:

① Resistors: $V = I Z_R$, where $Z_R = R$

② Capacitors: $V = I Z_C$, where $Z = \frac{-i}{\omega C}$

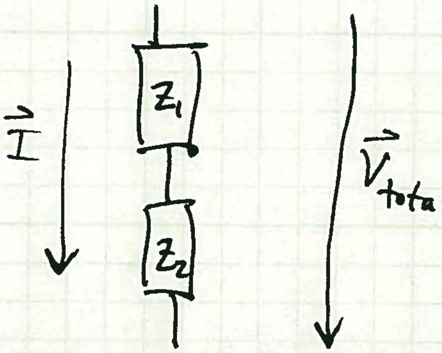
③ Inductors: $V = I Z_L$, where $Z_L = i\omega L$

phase shift between V & I

IF we combine two elements <u>in series</u>, the the <u>total voltage drop</u> is

$\vec{V}_{total} = \vec{I}Z_1 + \vec{I}Z_2$ ← because Z_1 & Z_2 share the same $\vec{I}$

$= \vec{I}(Z_1 + Z_2)$



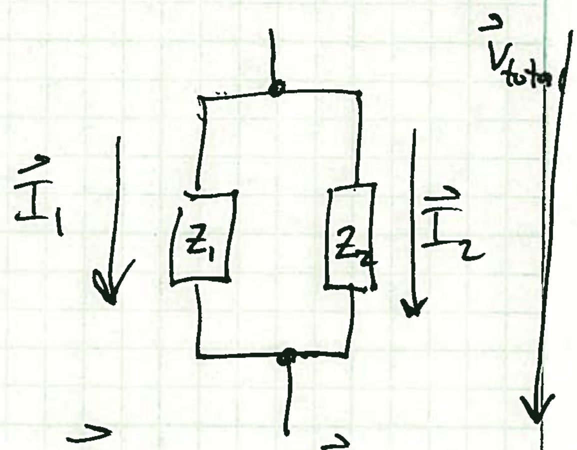
∴ $\vec{V}_{total} = \vec{I}(Z_{series})$ where $\boxed{Z_{series} = Z_1 + Z_2.}$

∴ The impedance of two elements in series is the simple sum of Z_1 & Z_2.

If we combine two elements <u>in parallel</u>, then the <u>total ~~current~~ voltage drop</u> is

$\vec{V}_{total} = \vec{I}_1 Z_1$ <u>and</u>

$\vec{V}_{total} = \vec{I}_2 Z_2$.



The total current is

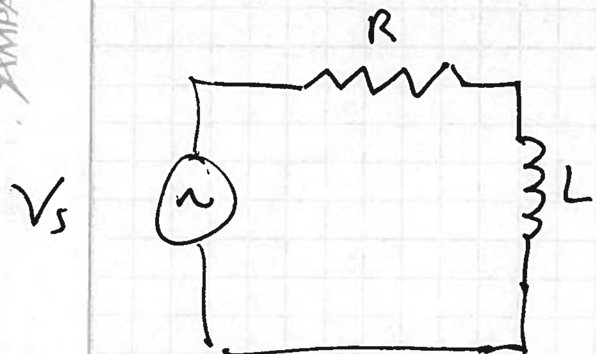
$\vec{I}_{total} = \vec{I}_1 \cancel{Z_1} + \vec{I}_2 \cancel{Z_2} = \frac{\vec{V}_{total}}{Z_1} + \frac{\vec{V}_{total}}{Z_2}$

∴ $\vec{V}_{total} = \vec{I}_{total}\left(\frac{1}{Z_1^{-1} + Z_2^{-1}}\right)$

$\vec{V}_{total} = \vec{I}_{total} Z_{parallel}$, where $\boxed{Z_{parallel} = \frac{1}{Z_1^{-1} + Z_2^{-1}}}$

$$\therefore \quad \boxed{\begin{aligned} Z_{series} &= Z_1 + Z_2 \\ Z_{parallel} &= \frac{1}{Z_1^{-1} + Z_2^{-1}} \end{aligned}}$$

Example RL ~~par.~~ Series circuit (driven)



$$Z_{total} = Z_R + Z_L$$

$$Z_{total} = R + i\omega L$$

$$\therefore \quad \vec{V}_s = \vec{I}_{total} \, Z_{total}$$

$$\vec{V}_s = \vec{I}_{total} \, (R + i\omega L)$$

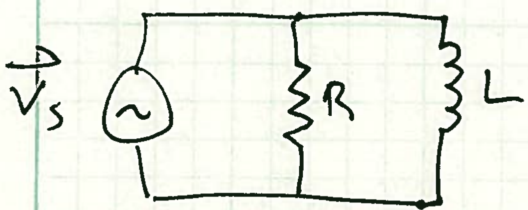
$\therefore$ phase difference between $\vec{V}_s$ & $\vec{I}_{total}$:

$$\delta = \text{phase of } (R + i\omega L) = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$\therefore$ magnitude of current:

$$|\vec{I}_{total}| = I_0 = \frac{\overbrace{|\vec{V}_s|}^{V_0}}{\underbrace{|R + i\omega L|}_{\sqrt{R^2 + (\omega L)^2}}} = \frac{V_0}{\sqrt{R^2 + (\omega L)^2}}$$

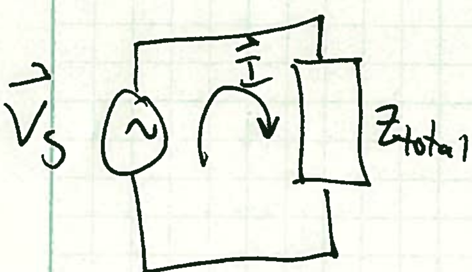
Example RL parallel circuit (driven)



$$Z_{total} = \frac{1}{Z_R^{-1} + Z_L^{-1}}$$

$$= \frac{1}{R^{-1} + (i\omega L)^{-1}}$$

or



$$= \frac{(R\omega L) \times (\omega L + iR)}{(\omega L - iR) \times (\omega L + iR)}$$

$$= \left(\frac{R\omega L^2}{(\omega L)^2 + R^2}\right)\left(\omega + i\frac{R}{L}\right)$$

$$Z_{total} = \left(\frac{R\omega L^2}{(\omega L)^2 + R^2}\right)(\omega + i\gamma)$$

$$\therefore \quad \vec{V}_{s\,\cancel{total}} = \vec{I}_{total}\left[\left(\frac{R\omega L^2}{(\omega L)^2 + R^2}\right)(\omega + i\gamma)\right] \qquad \text{where } \gamma \equiv \frac{R}{L}$$

Phase difference between $\vec{V}_s$ & $\vec{I}_{total}$:

$$\delta = \tan^{-1}\left(\frac{\gamma}{\omega}\right)$$

← For very high ω, $\delta \to 0$, like a resistor

← For very low ω, $\delta \to 90°$, like an inductor.

Transients

We've studied the steady-state behavior of harmonic oscillators $\Rightarrow$ how they behave once they settle down into a repeating pattern

Ex: Simple Harmonic Oscillator, no damping:

$$\ddot{x} + \omega_0^2 x = 0 \Rightarrow \text{Soln: } x(t) = A e^{i(\omega_0 t + \delta)},$$

A, δ determined by initial conditions

Ex: Simple Harmonic Oscillator with damping:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0 \Rightarrow \text{Soln: } x_d(t) = B e^{-\gamma t/2} e^{i(\omega_d t + \delta_d)}$$

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2/4}$$

B, δ_d determined by initial conditions. (Also, to recover the no-damping case, just set $\gamma = 0$.).

damped solution

Ex: Forced Oscillator with damping:

$$\ddot{x} + \gamma x + \omega_0^2 x = \frac{F_0}{m} e^{i\omega_F t}$$

$\uparrow$ forcing frequency

$$\text{Solution: } x_F(t) = A e^{i(\omega_F t + \delta_F)},$$

Forced solution

ω = forcing frequency

$$A(\omega_F) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_F^2)^2 + (\omega_F \gamma)^2}}, \quad \delta(\omega_F) = -\tan^{-1}\left[\frac{\omega_F \gamma}{(\omega_0^2 - \omega_F^2)}\right]$$

~~Why~~ Note: Our forced oscillator solution has no free parameters. Why not? ~~Does~~ Do the initial conditions play no role in the solution?

Answer: The initial conditions ~~determine the~~ play a role in the short-term behavior, but their influence or effect dies out as time goes forward. Our solution is the long term behavior only.

$\Rightarrow$ The long-term behavior does not depend on the initial conditions

$\Rightarrow$ The short term behavior does.

Question: How can we study the short term behavior of a forced oscillator?

~~Answer: Notice that our Eq. of Motion is linear:~~

$$\cancel{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}}$$

~~This means that if we find two different solutions, we can add them to get another solution:~~

~~Let $x_{Ann}(t)$ be Ann's solution~~

~~and $x_{Bob}(t)$ be Bob's solution.~~

<u>Answer</u>: Just add to our solution the related solution of a damped oscillator. (where $F_0 = \emptyset$).

$$x(t) = x_F(t) + x_d(t) \leftarrow \text{damped solution}$$

forced solution

$$= \underbrace{A(\omega_F)\, e^{i(\omega_F t + \delta_F)}}_{\substack{\text{long-term} \\ \text{behavior,} \\ \text{no free parameters}}} + \underbrace{B e^{-\gamma t/2}\, e^{i(\omega_d t + \delta_d)}}_{\substack{\text{short-term} \\ \text{behavior,} \\ \text{2 free parameters} \\ B \text{ \& } \delta_d.}}$$

"Short term solution" = "complementary solution" or "complementary function"

"Long-term solution" = "particular solution" = "transient solution"

But does our solution work? Try it:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x \stackrel{?}{=} \frac{F_0}{m} e^{i\omega t} \qquad \text{Substitute } x = x_F + x_d$$

$$(\ddot{x}_F + \ddot{x}_d) + \gamma(\dot{x}_F + \dot{x}_d) + \omega_0^2 (x_F + x_d) \stackrel{?}{=} \frac{F_0}{m} e^{i\omega t}$$

$$(\ddot{x}_d + \gamma \dot{x}_d + \omega_0^2 x_d) + (\ddot{x}_F + \gamma \dot{x}_F + \omega_0^2 x_F) \stackrel{?}{=} \frac{F_0}{m} e^{i\omega t}$$

But by definition, x_d satisfies $\ddot{x}_d + \gamma \dot{x}_d + \omega_0^2 x = \emptyset$

Therefore

$$\ddot{x}_F + \gamma \dot{x}_F + \omega_0^2 x_F \stackrel{?}{=} \frac{F_0}{m} e^{i\omega t} \qquad \text{Is this true?}$$

Answer: Yes! By definition

So our solution is

$$x(t) = A(\omega_F)\, e^{i(\omega_F t + \delta_F(\omega_F))} + B e^{-\gamma t/2} e^{i(\omega_d t + \delta_d)}$$

and its real part is

$$x(t) = A(\omega_F) \cos\left(\omega_F t + \delta_F(\omega_F)\right) + B e^{-\gamma t/2} \cos(\omega_d t + \delta_d)$$

where $A(\omega_F) = \dfrac{F_0/m}{\sqrt{(\omega_0^2 - \omega_F^2)^2 + (\omega_F \gamma)^2}}$, $\delta(\omega_F) = -\tan^{-1}\left[\dfrac{\omega_F \gamma}{\omega_0^2 - \omega_F^2}\right]$

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2/4}$$

and B and δ_d are determined by initial conditions

In general ~~once~~ we can find B & δ_d to satisfy any initial conditions, and the result is usually very complicated in the short term.

In the long term, the damping term dies out due to the $(e^{-\gamma t/2})$ Factor, and we are left with the steady state solution.

Simple Special Case.

Suppose we start with $x = 0$ and $\dot{x} = 0$ at $t = 0$, and we drive the oscillator at the resonant frequency : $\omega_F = \omega_0$.

Also, assume damping is very small, so

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2/4} \approx \omega_0 .$$

assume small compared to ω_0 (high Q assumption)

Then the solution is

$$A(\omega_f) = \boxed{A(\omega_0) = \frac{F_0}{m\omega_0\gamma}}$$

$$\tan(\delta(\omega_f)) = \frac{-\omega_0\gamma}{\omega_0^2 - \omega_0^2} \to -\infty, \quad \text{so } \boxed{\delta_f(\omega_0) = -\pi/2 .}$$

Also $\boxed{\omega_d \approx \omega_0}$

Then

$$x(t) = \frac{F_0}{m\omega_0\gamma}\underbrace{\cos(\omega_0 t - \pi/2)}_{\sin(\omega_0 t)} + Be^{-\gamma t/2}\cos(\omega_0 t + \delta_d)$$

$$x(t) = \frac{F_0}{m\omega_0\gamma}\sin(\omega_0 t) + Be^{-\gamma t/2}\cos(\omega_0 t + \delta_d)$$

$$x(t=0) = 0 = Be^{-\gamma t/2}\cos(\delta_d) \implies \boxed{\delta_d = \pi/2}$$

initial condition

And

$$\dot{x}(t) = \frac{F_0}{m\gamma}\cos(\omega_0 t) + B\left(-\omega_0\sin(\omega_0 t + \delta_d) - \frac{\gamma}{2}\right)e^{-\gamma t/2}\cos(\omega_0 t + \delta_d)$$

$$\dot{x}(t=0) = 0 = \frac{F_0}{m\gamma} + B\left(-\omega_0\sin(\delta_d) - \frac{\gamma}{2}\right)\underbrace{\cos(\delta_d)}_{\cos \pi/2}$$

Initial Condition

C

And

$$\dot{x}(t) = \frac{F_0}{m\gamma}\cos(\omega_0 t) + B\left(-\omega_0 \sin(\omega_0 t + \overset{\pi/2}{\delta_d})\right)e^{-\gamma t/2}$$
$$- \frac{B\gamma}{2}\left(\cos(\omega_0 t + \underset{\pi/2}{\delta_d})\right)e^{-\gamma t/2}$$

$$\dot{x}(t=0) = 0 = \frac{F_0}{m\gamma} + B\left(-\omega_0 \underbrace{\sin(\pi/2)}_{1}\right)$$

initial condition

$$- \frac{B\gamma}{2}\left(\underbrace{\cos(\pi/2)}_{0}\right)$$

$$0 = \frac{F_0}{m\gamma} - B\omega_0 \Rightarrow \boxed{B = \frac{F_0}{m\omega_0\gamma}}$$

Finally: when $\omega_f = \omega_0$, $\omega_d \approx \omega_0$, $x(t=0) = 0$ and $\dot{x}(t=0) = 0$

~~x(t)~~

$$x(t) = \frac{F_0}{m\omega_0\gamma}\sin(\omega_0 t) + \left(\frac{F_0}{m\omega_0\gamma}\right)e^{-\gamma t/2}\underbrace{\cos(\omega_0 t + \pi/2)}_{-\sin(\omega_0 t)}$$

$$\boxed{x(t) = \frac{F_0}{m\omega_0\gamma}\left(1 - e^{-\gamma t/2}\right)\sin(\omega_0 t)}$$

Special case:

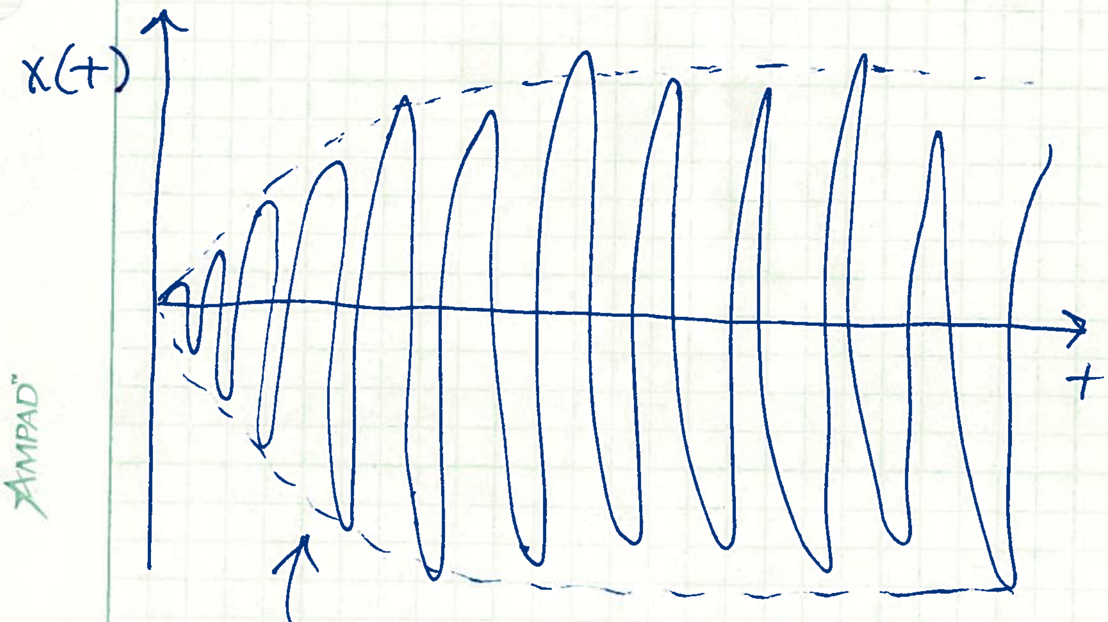
$\omega_f = \omega_0$

$\omega_d \approx \omega_0$ $(\gamma \ll \omega_0)$

$x = 0$ at $t = 0$

$\dot{x} = 0$ at $t = 0$

What does it look like?



exponential factor $\left(1 - e^{-\gamma t/2}\right)$

The amplitude increases then levels off to the steady state solution. This is like pushing a child on a swing in phase at the natural frequency of the swing.