We've studied:

- Simple Harmonic oscillator.

Properties:

a) Goes at its "natural frequency": $\omega_0 = \sqrt{k/m}$

b) ~~Amplitude~~ Frequency is independent of amplitude.

c) Amplitude and phase are determined by initial conditions.

- Forced oscillators with no damping: $F(t) = F_0 e^{i\omega t}$

Properties:

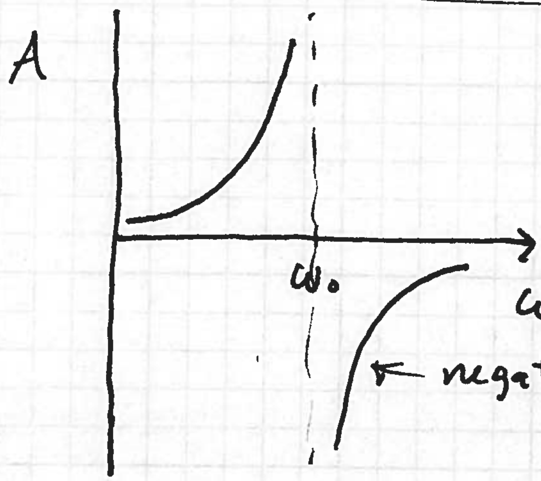
a) Oscillator goes at the forcing frequency.

b) Amplitude depends on how close the forcing frequency is to the natural frequency. Amplitude becomes very large if the forcing frequency is very close to ω_0.

c) ~~Phase~~ If $\omega < \omega_0$, oscillator is in phase with the external force $\Rightarrow$ no ~~phase shift~~ phase shift. If $\omega > \omega_0$, Oscillator ~~has~~ is 100% out-of-phase with the driving force: phase shift $= 180° = \pi$ radians

Forced Oscillator, no damping: $x(t) = A e^{i(\omega t + \delta)}$

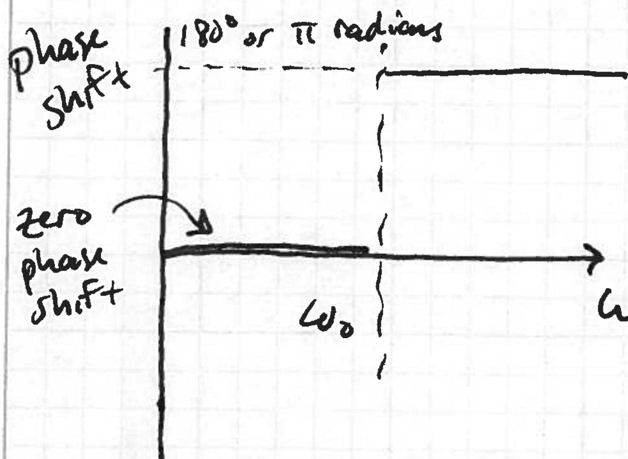
$\delta = \phi$, but A may be (+) or (−)



$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$\omega \leftarrow$ forcing frequency

← negative A means 180° phase shift.



$\omega \leftarrow$ forcing frequency

For the forced oscillator the initial condition determines transient behavior. We will study this later.

Now let's do:

Simple Harmonic oscillator with damping

Real mechanical oscillators always have some resistive force which is non-conservative. Resistive forces turn mechanical energy into heat, where it is usually lost. Resistive forces must be modeled empirically. A very simple model is

$$F_{resistive} = -bv$$

v ← velocity of oscillator

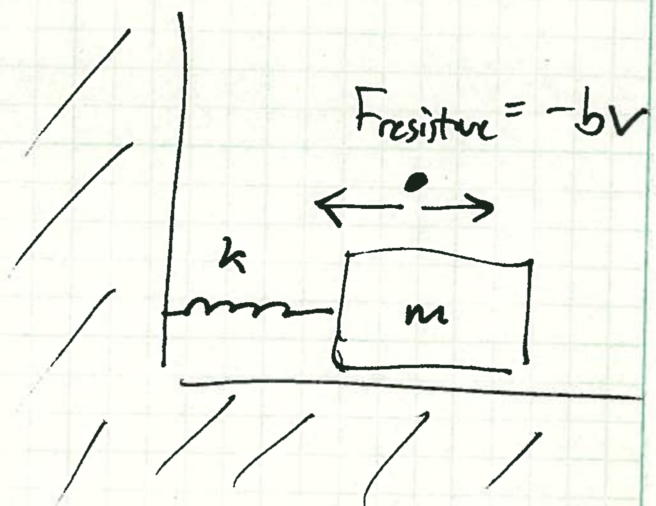
b ← constant

$-$ ← Force acts opposite the velocity

Resistive Forces

Real ~~oscillators~~ ~~may~~ may not be accurately modeled by such a simple force law, but this model allows the Eq. of Motion to be easily solved. Also, this model is often OK when v is not ~~so~~ too large.

Eq. of Motion:



~~$m\ddot{x}$~~

$$-kx - bv = m\ddot{x}$$

$$\uparrow$$

$$v = \dot{x}$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\uparrow$$

No driving force (for now)

Define $\gamma = \frac{b}{m}$. Then (Also $\omega_0^2 = k/m$)

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0}$$ Equation of Motion.

→ small γ means very little drag (resistive ~~resistance~~ force)

→ large γ means large drag.

Guessed Solution:

$$\boxed{x(t) = Ae^{i(\omega t + \delta)}}$$

⟹ A, δ, and ω are not yet known.

Substitute the guess:

$$\ddot{x} = -\omega^2 x \quad , \quad \dot{x} = i\omega x$$

Result: $-\omega^2 x + i\omega\gamma x + \omega_0^2 x = 0$

$x(t)$ divides out:

$$\boxed{-\omega^2 + i\omega\gamma + \omega_0^2 = 0} \leftarrow$$ Purely Algebraic Equation for ω in terms of γ and ω_0.

Note that $\boxed{\omega = \text{purely real}}$ cannot satisfy this equation. ~~So~~ Nor can ω = purely imaginary. So we must all ω to be complex, with both real and imaginary parts:

Let $\omega \equiv \omega_r + i\omega_i$, where ω_r = real and ω_i = ~~imaginary~~ real

Then our equation for ω says

$$-(\omega_r + i\omega_i)^2 + i(\omega_r + i\omega_i)\gamma + \omega_0^2 = 0$$

Two equations:

$$-\omega_r^2 + \omega_i^2 - \omega_i\gamma + \omega_0^2 = 0 \qquad (1)$$

and $$i(-2\omega_r\omega_i + \omega_r\gamma) = 0 \qquad (2)$$

From (2) we have $\boxed{\omega_i = \frac{\gamma}{2}}$.

Then substitute into (1):

$$\boxed{\omega_r^2 = \omega_0^2 - \frac{\gamma^2}{4}}$$

Therefore $\omega = \left(\omega_0^2 - \frac{\gamma^2}{4}\right) + i\left(\frac{\gamma}{2}\right)$.

Then our guessed solution says

$$x(t) = Ae^{i(\omega t+\delta)}$$
$$= Ae^{i((\omega_r+i\omega_i)t+\delta)}$$
$$= Ae^{-\omega_i t}\, e^{i(\omega_r t+\delta)}$$
$$\boxed{x(t) = Ae^{-\gamma t/2}\, e^{i(\omega_r t+\delta)}} \quad \text{where } \omega_r = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

To simplify notation, lets ~~just call~~ change the name of ω_r to just ω:

$$\boxed{\begin{gathered} x(t) = Ae^{-\gamma t/2}\, e^{i(\omega t+\delta)} \\ \text{where } \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \end{gathered}}$$

We still have 2 unknown constants: A & δ. These are determined by the initial conditions, just like the simple harmonic oscillator with no damping.

Comments:

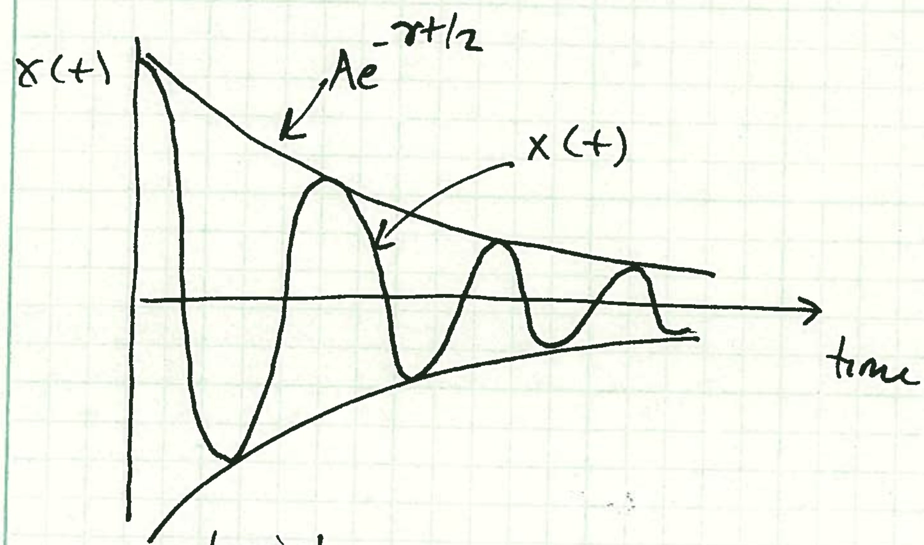
① If we set $\gamma = \emptyset$, we recover the SHO solution
$$x(t) = Ae^{i(\omega_0 t+\delta)}$$

② For non-zero γ, we have an oscillation at frequency $\omega = \underbrace{\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}$

not the natural frequency!

But if γ is small (small damping), then the frequency of oscillation is very close to ω_0.

(3) The amplitude of the oscillation decays exponentially in time:



The mechanical energy of the oscillator is being converted into heat by the drag force.

Mathematically, the exponential decay shows up when we realized that ω must be complex in order to satisfy the equation of motion.

Energy in Harmonic Oscillators

The "mechanical energy" is the sum of the kinetic and potential energy:

$E =$ mechanical energy $= KE + U$

For a mass on a spring, $U = \frac{1}{2}kx^2$

$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Now $x(t) = A\cos(\omega t+\delta)$ and $\dot{x}(t) = -A\omega\sin(\omega t+\delta)$

So $E(t) = \frac{1}{2}m\left(A^2\omega^2\sin^2(\omega t+\delta)\right) + \frac{1}{2}\underset{\uparrow\, m\omega_0^2}{k}\left(A^2\cos^2(\omega t+\delta)\right)$

Also $\omega_0^2 = k/m \Rightarrow k = m\omega_0^2$

$\cancel{\text{so } E(t) = \frac{1}{2}m\omega_0^2A^2\left(\sin^2(\omega t+\delta) + \cos^2(\omega t+\delta)\right)}$

$\cancel{E = \frac{1}{2}m\omega^2A^2 = \text{constant}}$

And in the case of <u>no damping</u>, $\underline{\omega = \omega_0}$

$$\therefore E(t) = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t+\delta) + \frac{1}{2}m\omega_0^2A^2\cos^2(\omega_0 t+\delta)$$

$$= \frac{1}{2}m\omega_0^2A^2\underbrace{\left(\sin^2(\omega_0 t+\delta) + \cos^2(\omega_0 t+\delta)\right)}_{1}$$

$$\boxed{E(t) = \tfrac{1}{2} m\omega_0^2 A^2 = \text{constant}}$$ (no damping).

Energy conservation (no damping).

But suppose the oscillator is "lightly damped"

Then $F_{drag} = -bv = -b\dot{x}$

$\uparrow$ small

and $\gamma = \frac{b}{m} \approx$ small.

The solution is

$$x(t) = \cancel{Ae^{-\gamma t/2}}\ \mathrm{Re}\left[Ae^{-\gamma t/2}e^{i(\omega t+\delta)}\right]$$

$$= Ae^{-\gamma t/2}\cos(\omega t + \delta)$$

when $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$.

For very small damping, γ^2 is very small,

so $\omega \approx \sqrt{\omega_0^2 - (\text{small})^2} \approx \omega_0$.

Then $x(t) = \underbrace{Ae^{-\gamma t/2}}_{A(t)}\cos(\omega_0 t + \delta)$ (light damping)

$$= A(t)\cos(\omega_0 t + \delta)$$

Then total mechanical energy

$$E = \tfrac{1}{2} m\omega_0^2 A^2 = \tfrac{1}{2} m\omega_0^2 (A(t))^2$$

$$= \frac{1}{2} m\omega_0^2 \left(A e^{-\gamma t/2}\right)^2$$

$$\boxed{E(t) = \underbrace{\tfrac{1}{2} m\omega_0^2 A^2}_{E_0 \text{ (lightly damped)}} e^{-\gamma t}}$$ ← total mechanical energy decays away exponentially

$$\boxed{E(t) = E_0 e^{-\gamma t}}$$ ⟹ energy is converted to heat by the drag force.

Quality Factor – Q

We want to define a quantity which tells us whether the oscillator loses energy quickly or slowly

- high "Q" = high quality = low energy loss
- low "Q" = low quality = high ~~[illegible]~~ energy loss

We define it this way:

Question: what fraction of the oscillator's energy is lost ~~[illegible]~~ in time $t = \frac{1}{\omega_0}$?

Answer: Fraction which remains is:

$$\frac{E(t)}{E_0} = e^{-\gamma t} \approx 1 - \gamma t + \ldots \quad \text{for small } t$$

fraction of energy which remains then $\approx 1 - \underbrace{\gamma t}$

must be the fraction that is lost.

Fraction lost in time t $= \gamma t$

$\boxed{\text{or}}$

Fraction lost in time $t = \frac{1}{\omega_0}$ $= \gamma\left(\frac{1}{\omega_0}\right) = \frac{\gamma}{\omega_0} \equiv \frac{1}{Q}$

"Quality Factor"

$\underbrace{Q \equiv \frac{\omega_0}{\gamma}}_{\text{unitless}} = \begin{cases} \text{very large, for very lightly damped oscillator} \\ . \end{cases}$

- If $Q = 100$, then damped oscillator loses $\approx \frac{1}{Q} \approx 1\%$ of its energy in time $t = \frac{1}{\omega_0}$.

- If $Q = 1000$, then damped oscillator loses $\approx \frac{1}{1000} = 0.1\%$ of its energy in time $t = \frac{1}{\omega_0}$

$$\boxed{Q = \frac{1}{\text{fraction of energy lost in time } t = \frac{1}{\omega_0}}}$$

or

$$\boxed{\begin{array}{c}\text{fraction of energy lost}\\ \text{in time } t = \frac{1}{\omega_0}\end{array} = \frac{1}{Q}}$$

Equivalently we can write:

T = period = time for one complete cycle

$$= \frac{1}{f} = \frac{2\pi}{\omega_0}$$

$\therefore$ fraction of energy lost in one period

$$= \gamma T$$

$$= \gamma\left(\frac{2\pi}{\omega_0}\right)$$

$$= \frac{2\pi}{(\omega_0/\gamma)}$$

$$= \frac{2\pi}{Q}$$

$$\boxed{Q = \frac{2\pi}{\text{fraction of energy lost in one period}}}$$

or

$$\boxed{\begin{array}{c}\text{fraction of energy}\\ \text{lost in one period}\end{array} = \frac{2\pi}{Q}}$$

We've studied:

Simple harmonic oscillator

Forced Harmonic Oscillator

Harmonic Oscillator with damping

$\boxed{\text{Forced Harmonic Oscillator with damping}}$ today

The Forced Oscillator with damping is the most general harmonic oscillator. The other three systems can be obtained by either damping to zero, setting the forcing function to zero, or both

Eq. of Motion: (Newton's 2nd Law)

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}$$

I've taken the liberty to ~~desing~~ choose $t = 0$ such that the forcing function is maximum at that time.

<u>Guessed Solution:</u> $x(t) = Ae^{i(\omega t+\delta)}$, as usual

The usual question: what are A, δ? [ω is the forcing frequency]

Substitute:

$$\ddot{x} = -\omega^2 x, \qquad \dot{x} = i\omega x$$

$$\Rightarrow \quad -\omega^2 x + i\omega\gamma x + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}$$

$$Ae^{i(\omega t+\delta)}$$

$e^{i\omega t}$ cancels everywhere

$\therefore$

$$\left[-\omega^2 + i\omega\gamma + \omega_0^2\right] Ae^{i\delta} = \frac{F_0}{m}$$

$$\underbrace{A(\omega_0^2-\omega^2) + i(\omega\gamma)A}_{\text{A complex \# in cartesian form}} = \underbrace{\frac{F_0}{m} e^{-i\delta}}_{\text{A complex \# in polar form}}$$

We have a real equation and an imaginary eq:

$$\boxed{A(\omega_0^2-\omega^2) = \frac{F_0}{m}\cos(-\delta) = \frac{F_0}{m}\cos(\delta)} \quad (1) \ \boxed{\text{real eq.}}$$

and

$$\boxed{A\omega\gamma = \frac{F_0}{m}\sin(-\delta) = -\frac{F_0}{m}\sin\delta} \quad (2) \ \boxed{\text{imaginary eq.}}$$

Ratio of the 2 equations eliminates A:

$$\frac{2}{1}: \quad \frac{\omega\gamma}{(\omega_0^2-\omega^2)} = -\frac{\sin(\delta)}{\cos(\delta)} = -\tan(\delta)$$

$$\delta = \text{phase shift} = \tan^{-1}\left[\frac{-\omega\gamma}{(\omega_0^2-\omega^2)}\right]$$

Phase shift of Forced Oscillator with damping.

$$\boxed{\underset{\text{driving frequency}}{\delta(\omega)} = -\tan^{-1}\left[\frac{\omega\gamma}{\omega_0^2-\omega^2}\right]}$$

Also we can take 1 & 2 and eliminate δ by squaring both equations and adding:

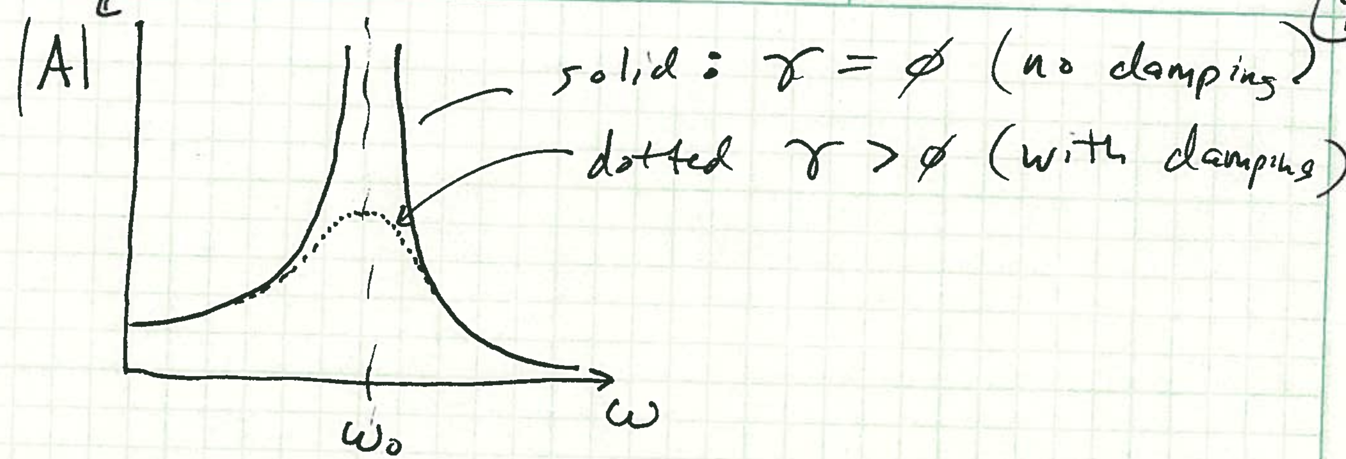
$$1^2 + 2^2: \quad A^2\left[(\omega_0^2-\omega^2)^2 - (\omega\gamma)^2\right] = \left(\frac{F_0}{m}\right)^2 \underbrace{(\cos^2\delta + \sin^2\delta)}_{1}$$

Amplitude of forced oscillator with damping, as a function of the driving frequency

$$\boxed{A(\underset{\text{driving frequency}}{\omega}) = \frac{\left(\frac{F_0}{m}\right)}{\sqrt{(\omega_0^2-\omega^2)^2 + (\omega\gamma)^2}}}$$

Just like the forced oscillator with no damping, forced oscillator with damping displays resonance: the amplitude of oscillation becomes large when $\omega \approx \omega_0$:

Absolute magnitude of A



An ideal oscillator with <u>no</u> <u>damping</u> has an <u>infinite amplitude</u> at $\omega = \omega_0$. But this never happens in nature, because there is always some damping.

Including the damping ($\gamma > \emptyset$), we see that the amplitude has a maximum when $\omega \simeq \omega_0$, but it is finite.

phase: Recall: we set $t = \emptyset$ so that the driving force is maximal at $t = \emptyset$. The oscillator has a <u>negative</u> <u>phase</u>, ~~with respect to the the~~ which means that it <u>lags</u> <u>behind</u> the driving force: