Summary of "Fourier's Trick"

Loaded String - N masses

Eigenvectors are given by

$$A_{pn} = \sin\left(\frac{pn\pi}{N+1}\right)$$

which mass

which normal mode

Or, in vector notation,

$$\vec{q}_n = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \ldots, \sin\left(\frac{Nn\pi}{N+1}\right)\right)$$

These normal mode vectors are <u>orthogonal</u>. We saw this on Homework #7 for the cases of $N = 2, 3,$ and 4. In general the orthogonality is mathematically guaranteed by the following trig identity

$$\vec{q}_n \cdot \vec{q}_m = \left(\sin\left(\frac{n\pi}{N+1}\right), \sin\left(\frac{2n\pi}{N+1}\right), \ldots\right)\cdot\left(\sin\left(\frac{m\pi}{N+1}\right), \sin\frac{2m\pi}{N+1}, \ldots\right)$$

$$= \sum_{j=1}^{N} \sin\left(\frac{jn\pi}{N+1}\right)\sin\left(\frac{jm\pi}{N+1}\right) = \left(\frac{N+1}{2}\right)\delta_{nm}$$

trig identity

(I'm not proving this, I'm just invoking a known trig identity.)

Just to be sure, let's put a box around it:

$$\boxed{\sum_{j=1}^{N} \sin\left(\frac{jn\pi}{N+1}\right)\sin\left(\frac{jm\pi}{N+1}\right) = \frac{(N+1)}{2}\delta_{nm}}$$

↑ orthogonality of the loaded string eigenvectors.

How does this compare to the continuous string?

Eigenvectors are continuous function of x:

$$\text{eigenvector } n = \sin\left(\frac{n\pi x}{L}\right)$$

↑ a continuous vector

They are orthogonal:

$$\boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{L}{2}\delta_{nm}}$$

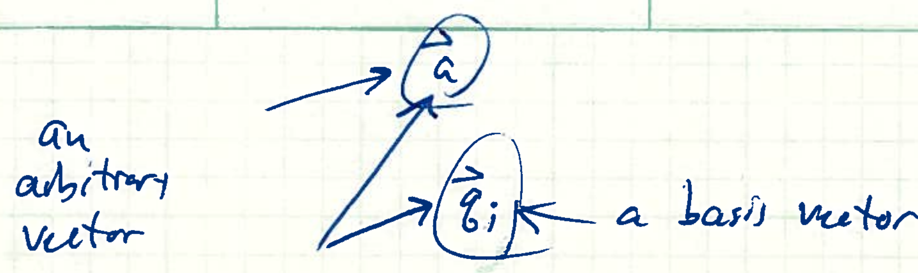
Compare to the discrete case:

$$\boxed{\sum_{j=1}^{N} \sin\left(\frac{jn\pi}{N+1}\right)\sin\left(\frac{jm\pi}{N+1}\right) = \frac{(N+1)}{2}\delta_{nm}}$$

both are dot products

Fourier's Trick:

To write an arbitrary vector as a sum of basis vectors, or normal modes, take the dot product with each basis vector:



The component of $\vec{a}$ in the direction of $\vec{q}_i$ is

$$\boxed{a_i = \frac{\vec{a}\cdot\vec{q}_i}{|\vec{q}_i|^2}}$$ Fourier's Trick.

In total, the complete vector $\vec{a}$ is the sum over all ~~the~~ basis vectors:

$$\vec{a} = a_1\vec{q}_1 + a_2\vec{q}_2 + \ldots$$

$$= \sum_i a_i\vec{q}_i \qquad \text{where} \quad a_i = \frac{\vec{a}\cdot\vec{q}_i}{|\vec{q}_i|^2}.$$

For the case of continuous vectors, like the normal modes of a continuous string, Fourier Trick says

$$a_n = \underbrace{\left(\frac{2}{L}\right)}_{\text{normalization factor.}} \underbrace{\int_0^L \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{basis vector}} \underbrace{y(x, t=0)}_{\text{initial condition}}\, dx}_{\text{integral is the dot product}}$$

For a loaded string, with finite eigenvectors, Fourier's Trick says

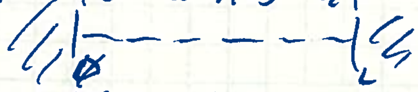
$$a_i = \frac{\overbrace{\vec{y}(t=0)}^{\text{initial condition}}\cdot\overbrace{\vec{q}_i}^{\text{basis vector}}}{\underbrace{|\vec{q}_i|^2}_{\text{normalization.}}}$$

Generalized Fourier Series

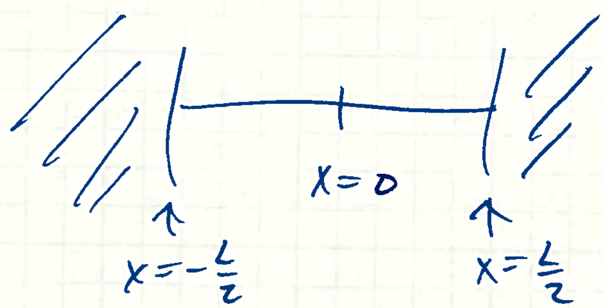
We've been studying a special type of Fourier Series called a "Fourier Sine Series"

$$y(x, t=0) = \sum_{n=1}^{\infty} a \sin\left(\frac{n\pi x}{L}\right)$$

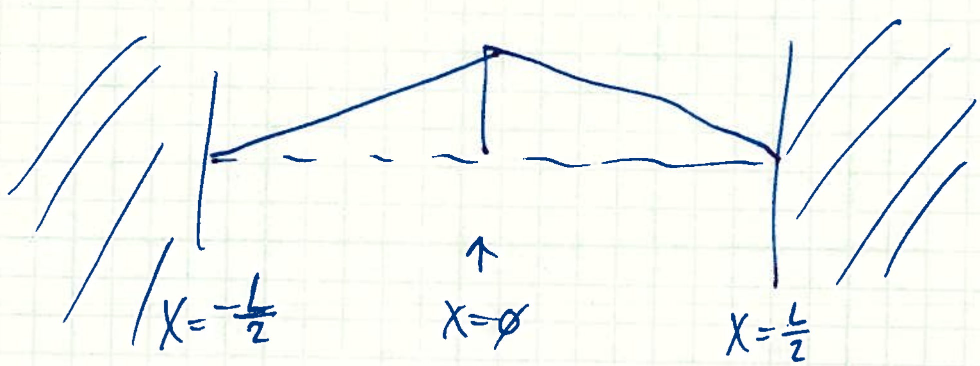
We like this series because it describes a string attached to walls at $x=0$ & $x=L$.



In general, however, we may choose to attach our string at other locations, like:



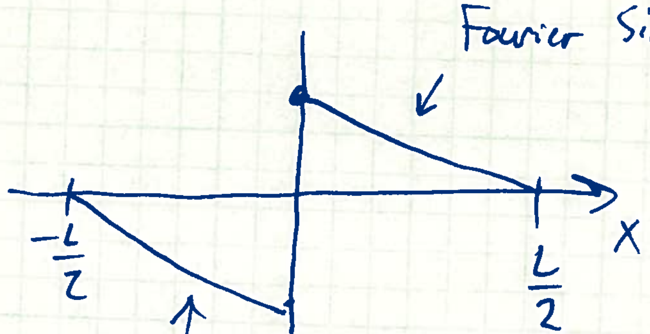
Ex Suppose, for example, that the initial shape of our string is a triangle, and our coordinate system is centered on the middle of the string:



Can we represent this shape as a sum of <u>sine</u> functions?

<u>Answer</u>: No, because sine functions are <u>odd</u> and this function is <u>even</u>. If we tried, we would get

Fourier Sine Series is OK here



but it gives the wrong sign here

It's because every term in the series is <u>odd</u>:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore f(-x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi(-x)}{L}\right) = -\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= -f(x) \leftarrow \text{odd function}$$

To represent an <u>even</u> function, we'll need a "Fourier Cosine Series"

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \leftarrow \text{"Fourier Cosine Series" for even functions.}$$

How do we determine the expansion coefficients $\{a_n\}$ for this series?

Answer: The cosine functions are orthogonal:

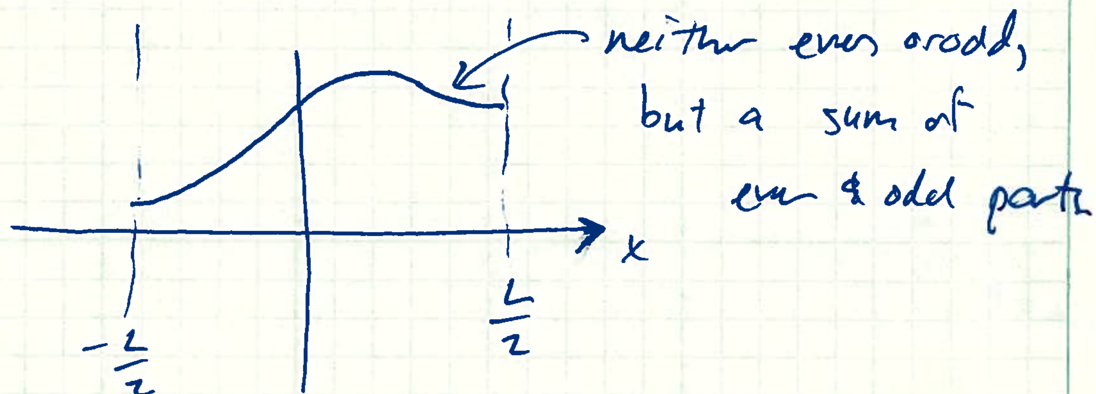
$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\,\delta_{nm}$$

Therefore Fourier's Trick works for them also:

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

In general, an arbitrary function is neither even or odd, but is a sum of even and odd parts:

$$f(x) = f_{odd}(x) + f_{even}(x)$$

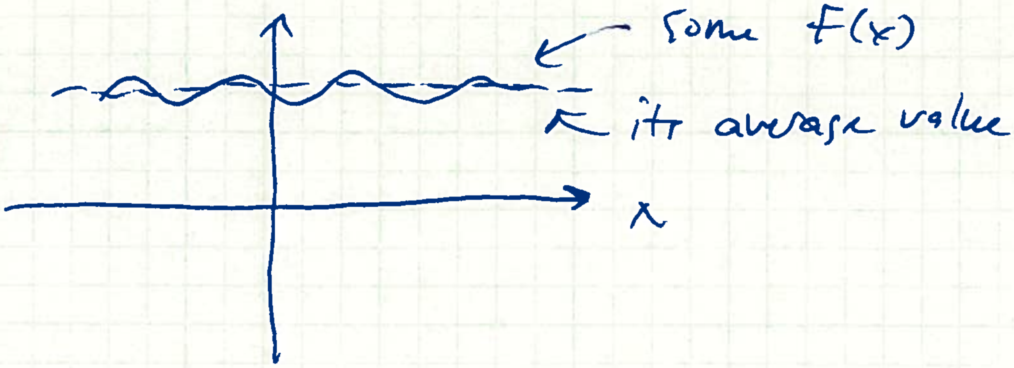


To represent a function like this, we need both Sine & Cosine Terms:

$$f(x) = \sum_n \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

But there's still one thing missing:

If the average value of the function is zero, then sines & cosines are fine. But if the function has a y-offset, then we have to add a constant:

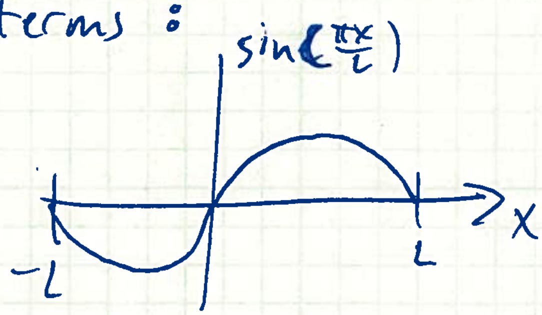


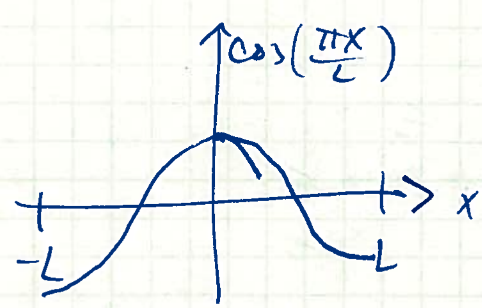
Finally, a complete, general Fourier Series is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_n \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]}$$

average value of $f(x)$

This series can represent almost any periodic function. But what is the period? Look at the $n=1$ terms:





The full period is $\boxed{2L}$.

Generalized Fourier Series:

f(x) is ① periodic with period 2L
② "square integrable" from $-L$ to L.

Then f(x) can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Note that normalization factor has changed because now we integrate over a distance of 2L instead of L.

Also, note that

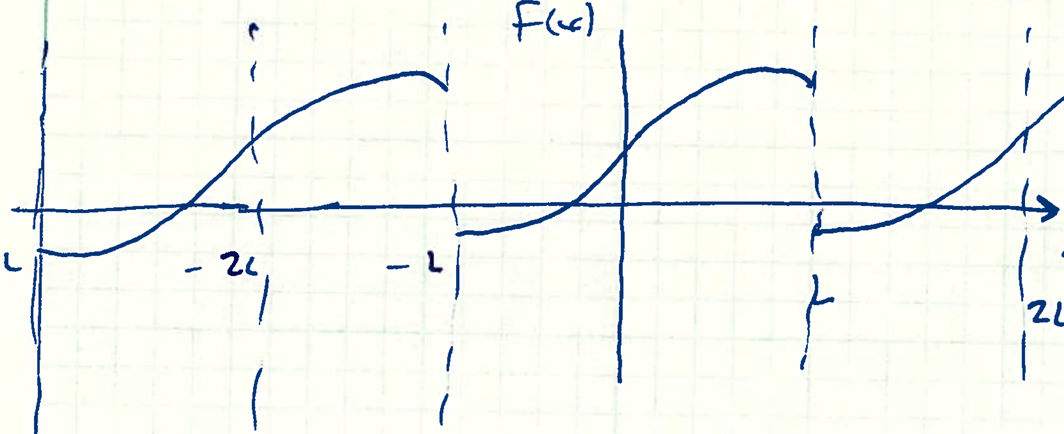
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{(0)\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

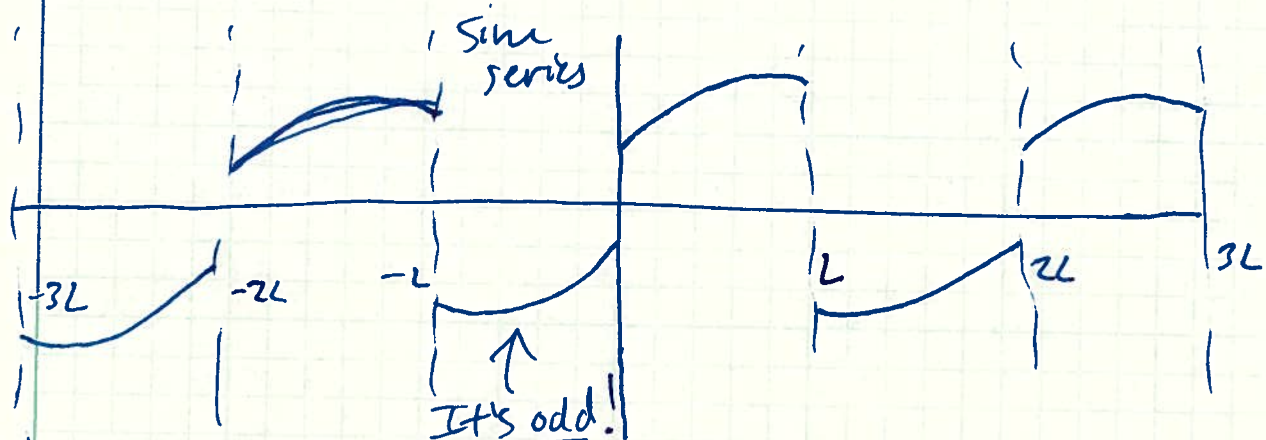
= 2 × average value of f(x) between $-L$ & L

Picture it:

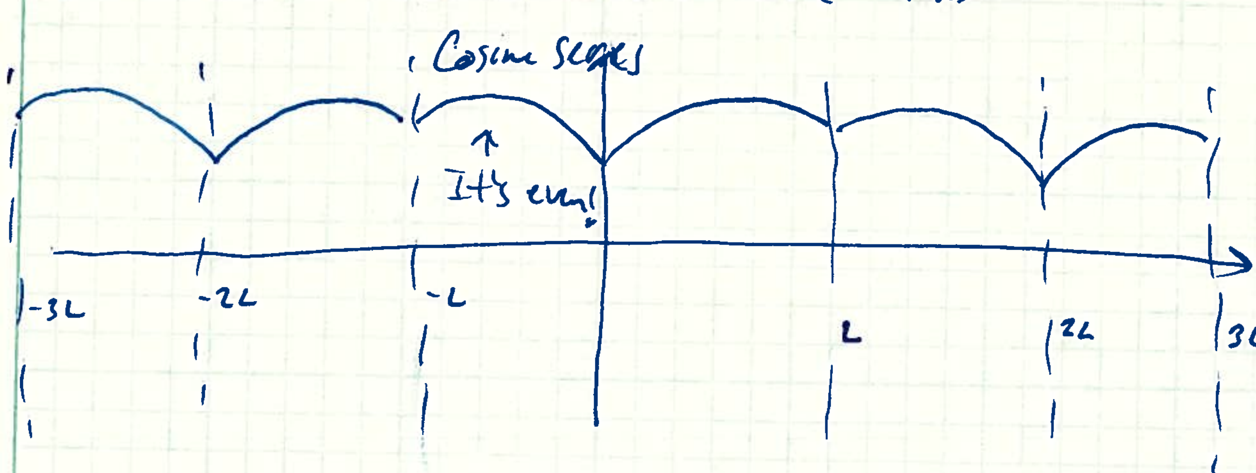
Suppose $F(x)$ looks like this:



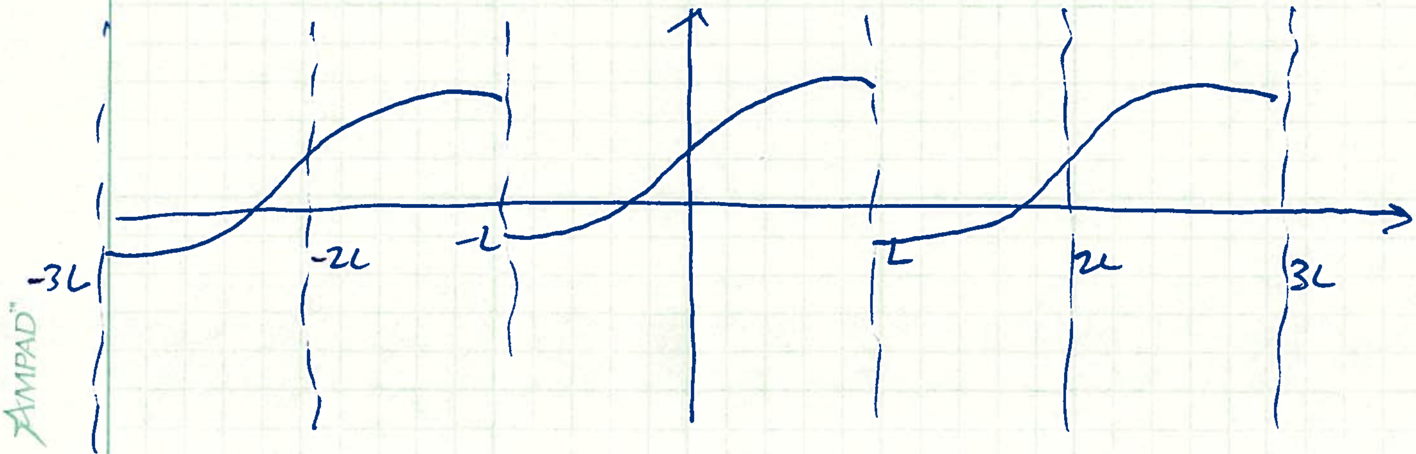
The sine series looks like this:



The cosine series looks like this

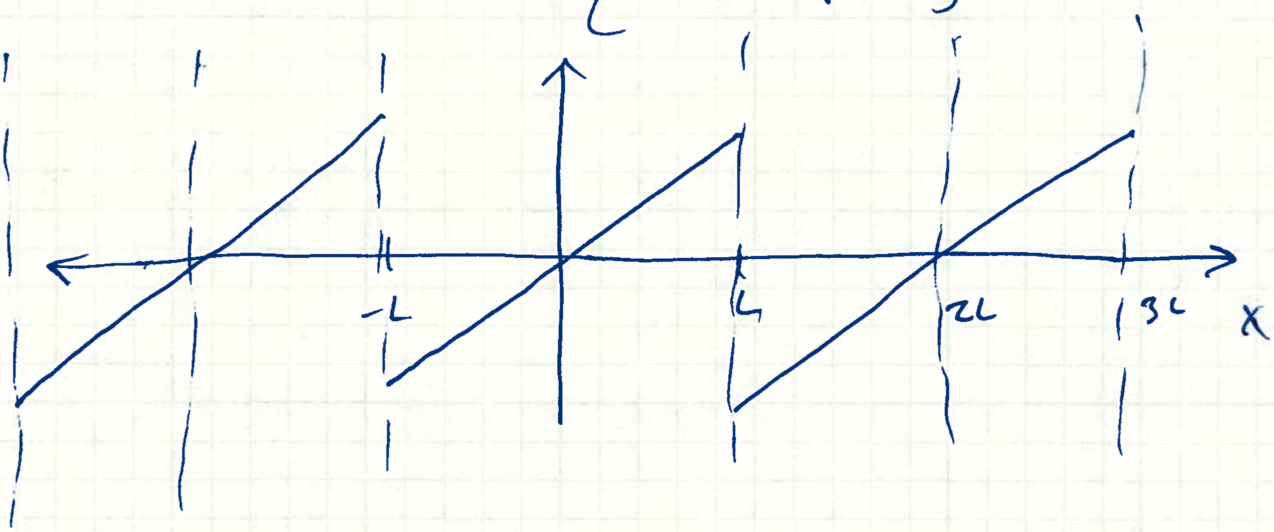


The complete, generalized Fourier Series looks exactly like $f(x)$



Example Calculation of a complete Fourier Series

Sawtooth: $f(x) = \begin{cases} x, \text{ for } -L < x < L \\ \text{and repeating} \end{cases}$



Fourier coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} x\,dx = \emptyset \leftarrow \text{average value is zero}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{x\cos\left(\frac{n\pi x}{L}\right)}_{\text{integrand is odd}} dx = \emptyset \leftarrow \text{no cosine terms!}$$

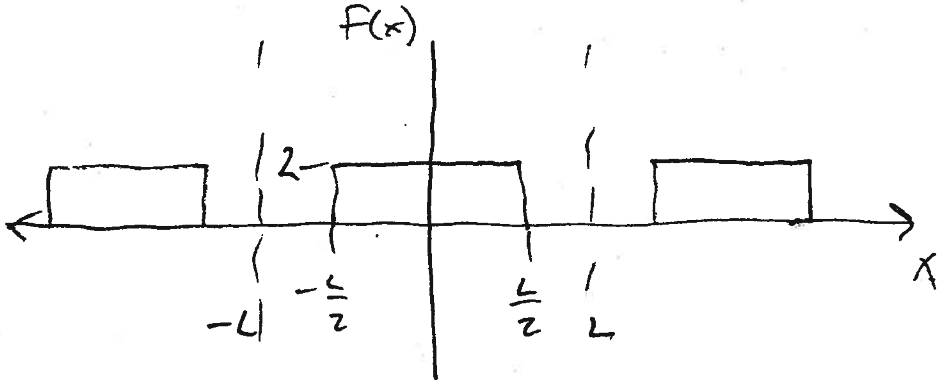
(Function is odd)

$$b_n = \underbrace{\frac{1}{L}\int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx}_{\substack{\text{look up this integral} \\ \text{or integrate by parts}}} = -2\left(\frac{L}{n\pi}\right)\cos(n\pi) + 2\left(\frac{L}{n\pi}\right)^2 \cancelto{\phi}{\sin(n\pi)}$$

$$\therefore \quad \boxed{b_n = \frac{2L}{n\pi}(-1)^{n+1}}$$

$$\therefore \quad f(x) = \frac{2L}{\pi}\sin\left(\frac{\pi x}{L}\right) - \frac{2L}{2\pi}\sin\left(\frac{2\pi x}{L}\right) + \frac{2L}{3\pi}\sin\left(\frac{3\pi x}{L}\right) + \ldots$$

Example: Another type of square wave:



$$f(x) = \begin{cases} 0, & -L < x < -\frac{L}{2} \\ 2, & -\frac{L}{2} < x < \frac{L}{2} \\ 0, & \frac{L}{2} < x < L \end{cases}$$ & repeating with period $2L$.

It's an even function of x, so the sine terms will be zero:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \emptyset$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2\,dx = 2 = 2\times \text{average value of } f(x)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \left(\frac{2}{L}\right)\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \left(\frac{2}{n\pi}\right)\left(\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right) = \left(\frac{4}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right) = \begin{cases} \emptyset, & n \text{ even} \\ \frac{4}{n\pi}(-1)^{(n-1)/2}, & n \text{ odd} \end{cases}$$

$$a_n = \left(\frac{4}{n\pi}\right)(-1)^{(n-1)/2} \quad \text{for } n \text{ odd only}$$

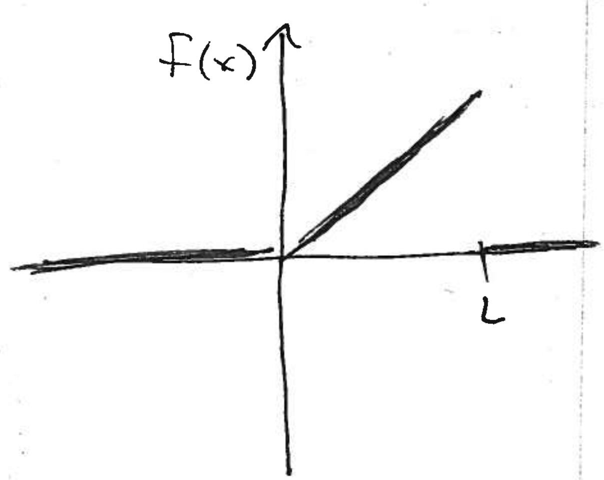
$$\therefore \quad f(x) = \underset{\downarrow \atop 1}{\frac{a_0}{2}} + \underset{\downarrow \atop \frac{4}{\pi}}{a_1} \cos\left(\frac{\pi x}{L}\right) + \underset{\downarrow \atop \emptyset}{a_2} \cos\left(\frac{2\pi x}{L}\right) + \underset{\downarrow \atop -\frac{4}{3\pi}}{a_3} \cos\left(\frac{3\pi x}{L}\right) + \cdots$$

$$= 1 + \frac{4}{\pi}\cos\left(\frac{\pi x}{L}\right) - \frac{4}{3\pi}\cos\left(\frac{3\pi x}{L}\right) + \cdots$$

$$= 1 + \sum_{n=\text{odd}}^{\infty} (-1)^{(n-1)/2}\left(\frac{4}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)$$

Consider this function

$$f(x) = \begin{cases} \emptyset, & x < \emptyset \\ x, & 0 \le x \le L \\ \emptyset, & x > L \end{cases}$$

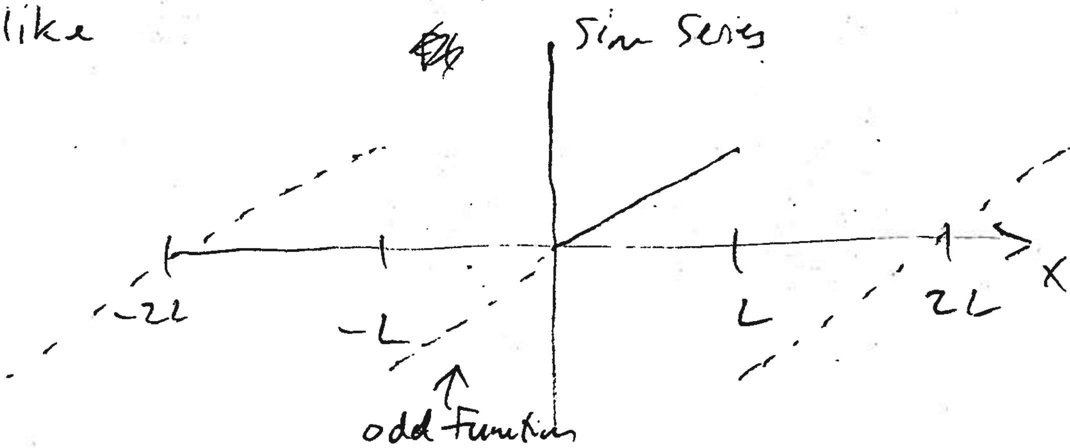


Suppose we wish to represent this function as a Fourier series in the interval $[0, L]$, and we don't care if the Fourier series gives $\emptyset$ outside the interval. So we would like to write

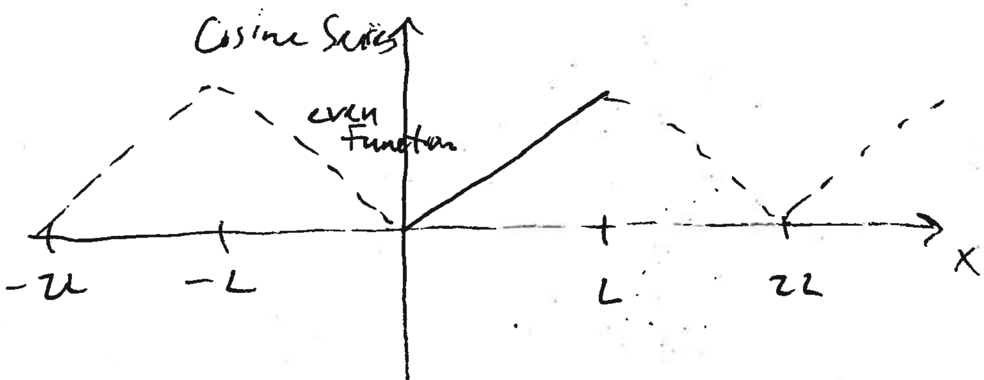
$$f(x) = \begin{cases} \emptyset, & x < \emptyset \\ \text{A Fourier Series}, & 0 \le x \le L \\ \emptyset, & x > L \end{cases}$$

Will we need sin terms, cosine terms, or both?

Answer: we can use either a Sine Series or a Cosine series. The Sine series will look like



The Cosine Series will look like



Between $x=0$ and $x=L$, either one can represent $f(x)$.

Sine Series Representation:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right) - \left(\frac{L}{n\pi}\right) x \cos\left(\frac{n\pi x}{L}\right)\right]$$

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$$= \frac{2}{L}\left[-\left(\frac{L^2}{n\pi}\right)\underbrace{\cos(n\pi)}_{(-1)^n}\right] = \boxed{\frac{2L}{(n\pi)}(-1)^{n+1}}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)}$$

Cosine Series Representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L}\int_0^L x\,dx = \left(\frac{2}{L}\right)\left(\frac{1}{2}L^2\right) = L = 2 \times \text{average value of } f(x)$$

$$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)x\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$= \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos(n\pi) - \left(\frac{L^2}{n\pi}\right)\right]$$

$$= \frac{2L}{(n\pi)^2}\underbrace{\left(\cos(n\pi) - 1\right)}_{-2,\ 0,\ -2,\ 0,\ \ldots}$$

$$\boxed{= \frac{-4L}{(n\pi)^2}}, \quad n = \text{odd only.}$$

$$\boxed{f(x) = \frac{L}{2} + \sum_{n=\text{odd}}^{\infty} \frac{-4L}{(n\pi)^2} \times \cos\left(\frac{n\pi x}{L}\right)}$$

So mathematically we can represent our function as either a Sine Series or Cosine Series, as long as we only care about the result between $x = 0$ and $x = L$.

But which series is <u>physically</u> <u>relevant</u> for a string connected to walls at <u>$x=0$</u> and <u>$x=L$</u>?

<u>Answer</u>: For this physical system, only the sine series is relevant, because the sine functions are the normal modes for this system. That means we can write the time evolution in a trivial way for the sine series:

$$\boxed{y(x,t) = \sum_{n=} (-1)^{n+1} \frac{2L}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}$$ ✓ correct

The cosine series is of no use to use for this system because the cosine terms do not satisfy the equation of motion and boundary condition:

$$y(x,t) \neq \boxed{\cancel{\frac{L}{2} + \sum_{n=odd}^{\infty} \frac{4L}{(n\pi)^2} (-1)^{(n+1)/2} \cos\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}}$$

Wrong!
Cosine terms are not normal modes!

Phys 273

Fourier Series in complex notation.

We've been using the trigonometric form of the Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \quad (1)$$

This is a long, complicated expression, and you have to do at least three integrals to find the expansion coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

So it would be nice to have a simpler, more compact way to write a Fourier Series. We can do that using complex exponentials:

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}} \quad (2)$$

← Completely equivalent to (1)!

!!

The first thing to note about the complex form is that the sum over (n) goes from $-\infty$ to $+\infty$, while the trigonometric form ~~goes from~~ has a sum that starts at 1 and goes to $+\infty$.

The relationship between 1 and 2 is the following:

To convert from the complex form to the trig form:

$$\boxed{\begin{aligned} a_n &= c_n + c_{(-n)} \\ b_n &= i(c_n - c_{(-n)}) \\ a_0 &= c_\emptyset + c_\emptyset = 2c_\emptyset \end{aligned}}$$

To convert from the trig form to the complex form:

Note that the $\{c_n\}$ are complex. real & imaginary parts.

$$c_n = \begin{cases} +\frac{1}{2}\left(a_{(-n)} + i b_{(-n)}\right), & \text{for } n < \emptyset \\ \frac{1}{2} a_0, & \text{for } n = \emptyset \\ \frac{1}{2}(a_n - i b_n), & \text{for } n > \emptyset \end{cases}$$

We can explicitly show that the two forms are equivalent using the above relations and Euler's Formula.

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Start with the trig form:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\boxed{\frac{1}{2}\left(e^{in\pi x/L} + e^{-in\pi x/L}\right)}} + b_n \overbrace{\sin\left(\frac{n\pi x}{L}\right)}^{\boxed{\frac{1}{2i}\left(e^{in\pi x/L} - e^{-in\pi x/L}\right)}}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{in\pi x/L} + \underbrace{\sum_{n=1}^{\infty} \frac{a_n}{2} e^{-in\pi x/L}}_{\text{call this term (A)}} + \sum_{n=1}^{\infty} \frac{b_n}{2i} e^{in\pi x/L} + \underbrace{\sum_{n=1}^{\infty} \frac{-b_n}{2i} e^{-in\pi x/L}}_{\text{call this term (B)}}$$

We can use a trick to re-write term (A) & term (B)

(A): $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{2} e^{-in\pi x/L}$ ← Let $\boxed{m \equiv -n}$

$$= \sum_{m=-1}^{-\infty} \frac{a_{(-m)}}{2} e^{im\pi x/L}$$

But m is just a dummy varicble, and we can re-name it if we like. So let's just convert m back to n: ($m \rightarrow n$)

(A): $\displaystyle\sum_{n=-1}^{-\infty} \frac{a_{(-n)}}{2} e^{in\pi x/L}$

Similiarly, term B can be writtn:

$$B: \sum_{n=-1}^{-\infty} \frac{-b_{(-n)}}{2i} e^{in\pi x/L}$$

So we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{in\pi x/L} + \sum_{n=-1}^{-\infty} \frac{a_{(-n)}}{2} e^{in\pi x/L}$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{2i} e^{in\pi x/L} + \sum_{n=-1}^{-\infty} \frac{-b_{(-n)}}{2i} e^{in\pi x/L}$$

Now substitute our conversion relations:

$$a_0 = 2c_0$$

$$a_n = c_n + c_{(-n)}$$

$$a_{(-n)} = c_{(-n)} + c_n = a_n$$

$$b_n = i\left(c_n - c_{(-n)}\right)$$

$$b_{(-n)} = -i\left(c_n - c_{(-n)}\right) = -b_n$$

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(\frac{c_n + c_{(-n)}}{2}\right) e^{in\pi x/L} + \sum_{n=1}^{-\infty} \left(\frac{c_n + c_{(-n)}}{2}\right) e^{in\pi x/L}$$

add $\updownarrow$ $\updownarrow$ cancel add $\updownarrow$ $\updownarrow$ cancel

$$+ \sum_{n=1}^{\infty} \left(\frac{c_n - c_{(-n)}}{2}\right) e^{in\pi x/L} + \sum_{n=-1}^{-\infty} \left(\frac{c_n - c_{(-n)}}{2}\right) e^{in\pi x/L}$$

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=-1}^{-\infty} c_n e^{in\pi x/L}$$

$$\downarrow$$

$$c_0 = c_0 \underbrace{e^{i(\emptyset)\pi x/L}}_{1}$$

So we have a term $c_n e^{in\pi x/L}$ for all n from $-\infty$ to $+\infty$, including $n = \emptyset$.

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}}$$

If we want to use this series to represent a particular function $f(x)$, we'll need to calculate the coefficients $\{c_n\}$. How can we do that?

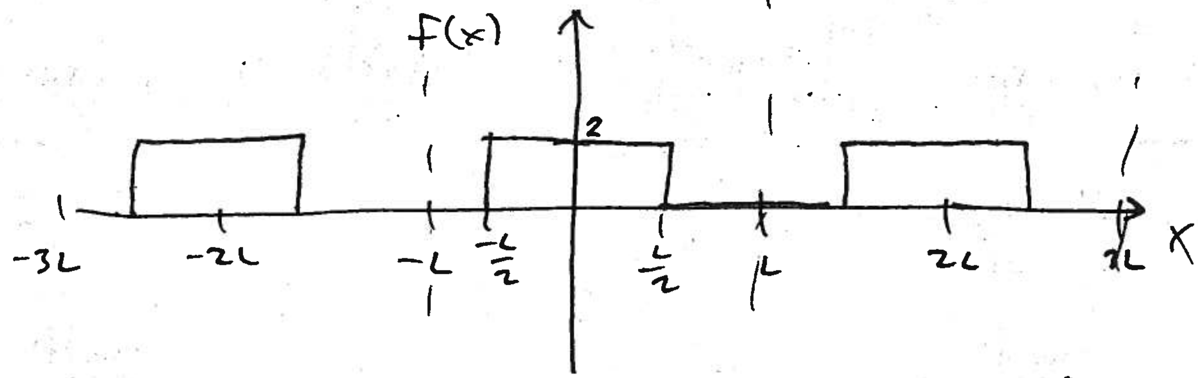
<u>Answer:</u> The basis functions $\{e^{in\pi x/L}\}$ are orthogonal:

$$\int_{-L}^{L} \left(e^{in\pi x/L}\right)\left(e^{-im\pi x/L}\right) dx = \begin{cases} 2L, & n = m \\ \emptyset, & n \neq m \end{cases} = 2L\,\delta_{nm}$$

Therefore Fourier's Trick will work:

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\, dx}$$

Example Square wave:



We've already calculated the trig series for this function. (In class on Thursday). Result:

$$a_0 = 2$$

$$a_n = \frac{4}{n\pi}(-1)^{(n-1)/2}, \text{ odd } n \text{ only (even } n \text{ is zero)}.$$

$$b_n = \emptyset. \leftarrow \text{Sine terms are zero.}$$

The complex calculation is:

$$c_0 = \frac{1}{2L}\int_{-L}^{L} F(x)\underbrace{e^{-i(\emptyset)\pi x/L}}_{1}\,dx = \frac{1}{2L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2\,dx = \boxed{1}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$$

$$= \frac{1}{2L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2e^{-in\pi x/L}\,dx$$

$$= \frac{1}{L}\left[\left(\frac{-L}{in\pi}\right) e^{-in\pi x/L}\right]\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$c_n = \left(\frac{-1}{in\pi}\right)\left(e^{-in\pi/2} - e^{in\pi/2}\right)$$

$$= \frac{2}{n\pi}\underbrace{\left(\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right)}_{\sin\left(\frac{n\pi}{2}\right)}$$

$$\boxed{c_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$$ ← for $n \neq \emptyset$. $(n > \emptyset$ & $n < \emptyset)$.

Note that this is not valid for $n=\emptyset$ because we would divide by zero.

Is our result the same as our trig calculation?

Check it:

$$a_n = c_n + c_{(-n)} = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{(-n\pi)}\sin\left(-\frac{n\pi}{2}\right)$$

$$= \frac{4}{n\pi}\underbrace{\sin\frac{n\pi}{2}}_{(-1)^{(n-1)/2}}$$ for odd n

$$= \underbrace{\frac{4}{n\pi}(-1)^{(n-1)/2}}$$ For odd n

Yes, this is the same result as before.

Also $a_0 = 2c_0$ as expected.

Also check the b_n:

$$b_n = i\left(c_n - c_{(-n)}\right) = i\left(\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{(-n\pi)}\sin\left(\frac{-n\pi}{2}\right)\right)$$

$$\boxed{= 0}$$

Yes this is the same as before. There are no sine terms because the function is <u>even</u>.

<u>Summary</u>: Two ways to write this Fourier Series

$$\boxed{f(x) = 1 + \sum_{\substack{n=1,3,5\ldots \\ \text{(odd } n \text{ only)}}}^{\infty} (-1)^{(n-1)/2} \left(\frac{4}{n\pi}\right) \cos\left(\frac{n\pi x}{L}\right)}$$

AND

$$\cancel{f(x) = 1 + \sum_{\substack{n=-\infty \\ (\text{except } n=0)}}^{\infty} \left(\frac{4}{n\pi}\right) (-1)^{(n-1)/2}}$$

$$\boxed{f(x) = 1 + \sum_{n=-\infty}^{\infty} (-1)^{(n-1)/2} \left(\frac{2}{n\pi}\right) e^{in\pi x/L}}$$

the $n=0$ term

except $n=0$ is excluded