Fourier's Trick relies upon the following mathematical identity:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

We like to write this more compactly:

Define $\boxed{\delta_{mn} \equiv \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}}$ "Kronecker Delta"

Thus $\delta_{11} = 1$, $\delta_{12} = \emptyset$, $\delta_{13} = \emptyset$, $\delta_{22} = 1$, ect

Using the Kronecker Delta we can say

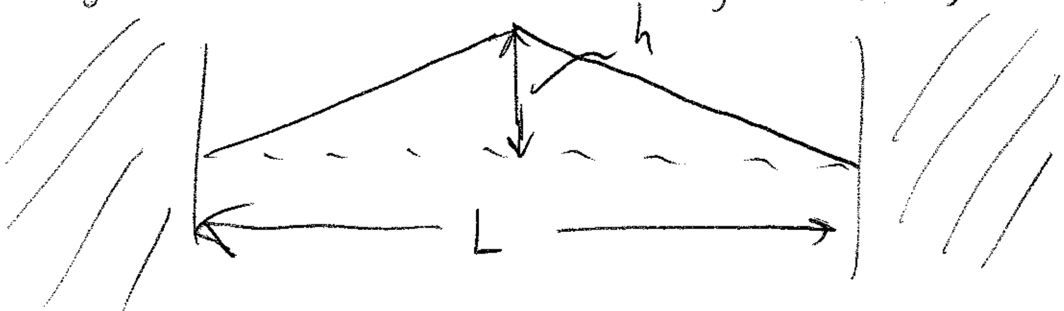
$$\boxed{\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2}\delta_{mn}}$$

or

$$\boxed{\frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \delta_{mn}}$$

---

Let's go back to our triangular string:



This is the shape at $t=\emptyset$. The functional form is

$$y(x,t=\emptyset) = \begin{cases} \left(\frac{2h}{L}\right) x & , \quad 0 \le x \le \frac{L}{2} \\ \left(\frac{2h}{L}\right)(L-x) & , \quad \frac{L}{2} \le x \le L \end{cases}$$

We want to describe this simple function in a much more complicated way: as an infinite sum of normal modes:

$$y(x,t=\emptyset) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

The question is: what are the $\{a_n\}$?

Fourier's Trick tells us that any particular coefficient, ~~can be calculated~~ for example, the $m^{th}$ coefficient ($a_m$), can be calculated by evaluating this integral:

$$\boxed{a_m = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) y(x)\, dx}$$

For our function $y(x)$, this integral is

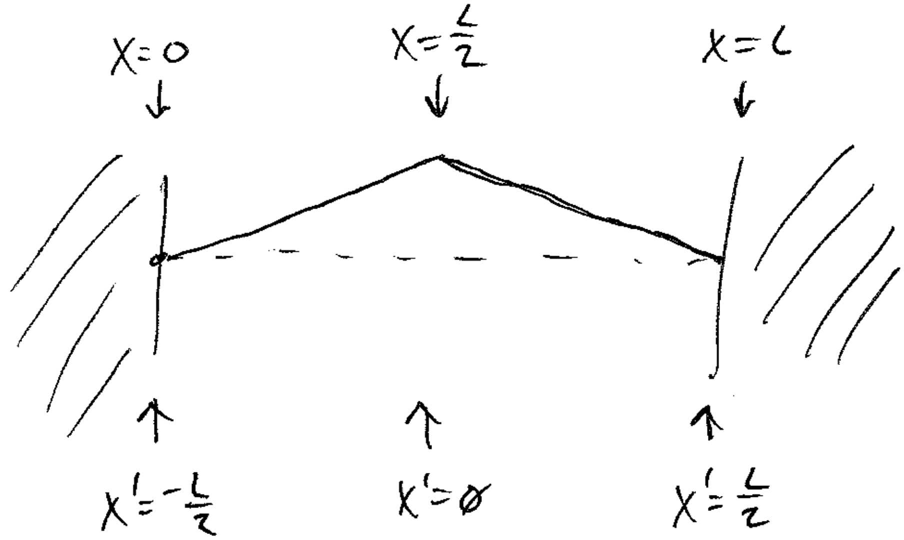
$$a_m = \frac{2}{L}\int_0^{L/2} \sin\left(\frac{m\pi x}{L}\right)\left(\frac{2hx}{L}\right)dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} \sin\left(\frac{m\pi x}{L}\right)\left(\frac{2h(L-x)}{L}\right)dx$$

It turns out that the easiest way to evaluate this integral is to move our coordinate system -

$$\text{Let} \quad x' \equiv x - \frac{L}{2}$$

$$\text{so that} \quad x = x' + \frac{L}{2}$$

This means that $x'=\emptyset$ is the center of the string




In terms of $x'$, our string position at $t=\emptyset$ is

$$\boxed{y(x', t=\emptyset) = \begin{cases} \left(\frac{2h}{L}\right)\left(x'+\frac{L}{2}\right), & -\frac{L}{2} \le x' \le \emptyset \\ \left(\frac{2h}{L}\right)\left(-x'+\frac{L}{2}\right), & 0 \le x' \le \frac{L}{2} \end{cases}}$$

Note that y is an even function of $x'$.

Also, we have the following math theorem:

If $x = x' + \frac{L}{2}$,

Then $$\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} (-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right), & \text{for } m=\text{odd} \\ (-1)^{m/2} \sin\left(\frac{m\pi x'}{L}\right), & \text{for } m=\text{even} \end{cases}$$

Now our integral has 2 cases:

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right) dx' \quad \text{for } m = \text{odd}$$

AND

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')\,(-1)^{(m)/2} \sin\left(\frac{m\pi x'}{L}\right)} dx' \quad \text{for } m = \text{even}.$$

This integrand is an odd function of $x'$, because $y(x')$ is even, and $\sin\left(\frac{m\pi x'}{L}\right)$ is odd.

Therefore the integral is zero because we integrate from $-\frac{L}{2}$ to $\frac{L}{2}$.

So we only need to evaluate the case for $m = \text{odd}$:

$$a_m = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underbrace{y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right)} dx' \quad , \ m = \text{odd}.$$

This integrand is even because $y(x')$ and $\cos\left(\frac{m\pi x'}{L}\right)$ are both even functions of $x'$. Since we integrate from $-\frac{L}{2}$ to $\frac{L}{2}$, we can just integrate from zero to $\frac{L}{2}$ and multiply by 2:

$$a_m = (2)\frac{2}{L}\int_{0}^{\frac{L}{2}} y(x')\,(-1)^{(m-1)/2} \cos\left(\frac{m\pi x'}{L}\right) dx'$$

$$a_m = (2)\left(\frac{2}{L}\right)(-1)^{(m-1)/2}\left(\frac{2h}{L}\right)\int_0^{\frac{L}{2}}\left(-x' + \frac{L}{2}\right)\cos\left(\frac{m\pi x'}{L}\right)dx'$$

$$= \left(\frac{8h}{L^2}\right)(-1)^{(m-1)/2}\left[\left(-\left(\frac{L}{m\pi}\right)^2\cos\left(\frac{m\pi x'}{L}\right) - \frac{x'L}{m\pi}\sin\left(\frac{m\pi x'}{L}\right) + \left(\frac{L}{2}\right)\left(\frac{L}{m\pi}\right)\sin\frac{m\pi x'}{L}\right)\Bigg|_0^{\frac{L}{2}}\right]$$

cancel

$$= \left(\frac{8h}{L^2}\right)(-1)^{(m-1)/2}\left[-\left(\frac{L}{m\pi}\right)^2\overbrace{\cos\left(\frac{m\pi}{2}\right)}^{\text{zero for } m=\text{odd}} - \frac{L^2}{2m\pi}\sin\left(\frac{m\pi}{2}\right) + \left(\frac{L^2}{2m\pi}\right)\sin\left(\frac{m\pi}{2}\right) - \left(-\left(\frac{L}{m\pi}\right)^2\right)\right]$$

$$\boxed{a_m = \frac{8h}{(m\pi)^2}(-1)^{(m-1)/2}, \quad m = \text{odd}} \quad \text{and} \quad \boxed{a_m = 0 \text{ for } m \text{ even}}$$

Therefore

$$a_1 = \frac{8h}{\pi^2}$$

$$a_2 = 0$$

$$a_3 = \frac{-8h}{9\pi^2}$$

$$a_4 = 0$$

$$a_5 = \frac{8h}{25\pi^2}$$

$$\vdots$$

Or we can write

$$y(x, t=\phi) = \frac{8h}{\pi^2}\sin\left(\frac{\pi x}{L}\right) - \frac{8h}{9\pi^2}\sin\left(\frac{3\pi x}{L}\right) + \frac{8h}{25\pi^2}\sin\left(\frac{5\pi x}{L}\right) + \ldots$$

## Why did we do this?

Recall our motivation: The general solution to the wave equation is a sum over normal modes:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} \implies y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$$

where $c_n$ = real and imaginary
$\equiv a_n + i b_n$

Given the initial condition:

$$y(x, t=0) = \text{a triangle} = \begin{cases} \left(\frac{2h}{L}\right) x, & 0 \le x \le \frac{L}{2} \\ \frac{2h}{L}(L-x), & \frac{L}{2} \le x \le L \end{cases}$$

We found the $a_n$:

$$a_n = \begin{cases} \left(\frac{8h}{(n\pi)^2}\right)(-1)^{(n-1)/2} & \text{for } n = \text{odd} \\ \phi & \text{for } n = \text{even} \end{cases}$$

What about the imaginary part, $\{b_n\}$?
It is determined by the <u>initial</u> <u>velocity</u>:

$$\dot{y}(x, t=0) = \sum_{n=1}^{\infty} -\omega_n b_n \sin\left(\frac{n\pi x}{L}\right)$$

If we release the string from rest, then we must have $\dot{y}(x, t=0) = 0 \Rightarrow \boxed{b_n = 0 \text{ for all } n}$

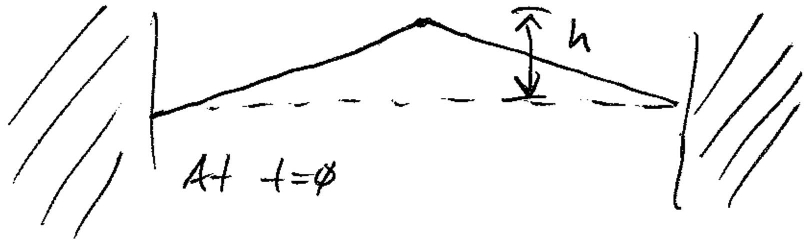
So our final, time-dependent solution is

$$\boxed{y(x,t) = \sum_{\substack{n=1 \\ (\text{only odd}(n))}}^{\infty} \frac{8h}{(n\pi)^2} (-1)^{\left(\frac{n-1}{2}\right)} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}} \quad \boxed{\text{odd } n \text{ only!}}$$

where $\omega_n = \sqrt{\frac{T}{\rho}} \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

We write the initial condition function $y(x, t=0)$ as a sum over normal modes because then the time development is extremely simple: each normal mode goes forward in time with its own harmonic factor $\left(e^{i\omega_n t}\right)$.

We did an example: the triangular string:




How does the string evolve in time? Answer:

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t} \quad , \quad \omega_n = \sqrt{\frac{T}{\rho}}\,\frac{n\pi}{L}$$

And what are the $\{a_n\}$ for the triangular initial conditions? Answer

$$a_n = \frac{2}{L}\int_{\phi}^{L} y(x,t=\phi)\,\sin\left(\frac{n\pi x}{L}\right)dx$$

We calculated this integral on Thursday and found

$$a_n = \begin{cases} \dfrac{8h}{(n\pi)^2}(-1)^{(n-1)/2} & , \text{ for } n = \text{odd} \\ \phi & , \text{ for } n = \text{even} \end{cases}$$

Therefore, at $t=\phi$,

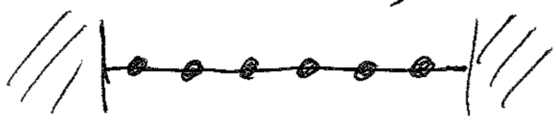
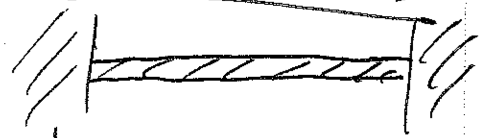
$$y(x,t=\phi) = \sum_{\substack{n=1,\\ \text{odd}\\ (n)\text{ only!}}}^{\infty} \underbrace{\frac{8h}{(n\pi)^2}(-1)^{(n-1)/2}}_{a_n} \sin\left(\frac{n\pi x}{L}\right) = \text{a triangle.}$$

Here we have succeeded in writing a very simple function, a triangle, as an extremely complicated <u>infinite</u> sum of sine functions. Why did we do that?

Answer: Because once we've written the initial condition as a sum over normal modes, now we can get the time dependence in a trivial way: Just tack-on the exponential phase factor to each term in the sum:

$$\boxed{y(x,t) = \sum_{\substack{n=1, \\ \text{odd}(n) \\ \text{only}}}^{\infty} \frac{8h}{(n\pi)^2} (-1)^{(n-1)/2} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}$$

## Comparison of the loaded string with the continuous string.

| | Loaded String | Continuous String |
|---|---|---|
| Diagram: |  |  |
| Eq. of Motion: | $\ddot{y}_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$ | $\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$ |
| Nature: | Discrete (N masses) | Continuous (infinite # of masses) |
| Number of Normal Modes: | N | infinite |
| Normal Mode Amplitude Relationship: | $y_{pn} = C_n \sin\left(\frac{pn\pi}{N+1}\right)$ | $y_n(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$ |
| Normal Frequencies: | $\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$ | $\omega_n = \sqrt{\frac{T}{\rho}}\ \frac{n\pi}{L}$ |
| General Solution: | $y_p = \sum_{n=1}^{\infty} C_n \sin\left(\frac{pn\pi}{N+1}\right) e^{i\omega_n t}$ | $y_n(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}$ |
| | initial conditions | initial conditions. |

## How to determine the $\{C_n\}$: Initial Conditions and Fourier's Trick.

For any set of initial conditions, we must find the coefficients $\{C_n\}$ which satisfy those initial conditions. Once the $\{C_n\}$ have been determined, then the time development for all time is given by the general solution.

## Recap of Fourier's Trick

For the loaded string, we have orthogonal eigenvectors:

$$\underbrace{\vec{q}_n \cdot \vec{q}_m = \sum_{p=1}^{N} \sin\left(\frac{pn\pi}{N+1}\right) \sin\left(\frac{pm\pi}{N+1}\right)}_{\text{a dot product of discrete vectors}} = \left(\frac{N+1}{2}\right) \delta_{nm}$$

$\delta_{nm}$ ← Kronecker Delta:

$$\delta_{nm} = \begin{cases} 1, & n=m \\ \emptyset, & n \neq m \end{cases}$$

For the continuous string, we also have orthogonal eigenvectors:

$$\underbrace{\int_{\emptyset}^{L} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{eigenvector}} \underbrace{\sin\left(\frac{m\pi x}{L}\right)}_{\text{eigenvector}} dx}_{\text{a dot product of continuous vectors}} = \frac{L}{2} \underbrace{\delta_{nm}}_{\text{they are orthogonal}}$$

~~Thus~~ The property of orthogonality makes Fourier's Trick work:

For the loaded string: Let $\vec{y}_0$ be the vector of initial conditions. Then

$$\vec{y}_0 \cdot \vec{q}_m = \left(\sum_{n=1}^{N} \left(a_n \vec{q}_n\right)\right) \cdot \vec{q}_m = \sum_{n=1}^{N} a_n \underbrace{\vec{q}_n \cdot \vec{q}_m}_{\left(\frac{N+1}{2}\right)\delta_{nm}} = \sum_{n=1}^{N} a_n \left(\frac{N+1}{2}\right) \delta_{nm}$$

$\delta_{nm}$ kills all terms in the sum except $n=m$.

$$= a_m\left(\frac{N+1}{2}\right) = a_m\left(\vec{q}_m\cdot\vec{q}_m\right) = a_m|\vec{q}_m|^2$$

initial conditions

$$\therefore \boxed{a_m = \frac{\vec{y}_0\cdot\vec{q}_m}{|\vec{q}_m|^2}}$$ Fourier's Trick

expansion coefficient

normalization

For the continuous string the same reasoning applies. Let $y(x, t=\emptyset) = y(x)$ be the initial condition. Then

$$\underbrace{\int_0^L y(x)\sin\left(\frac{m\pi x}{L}\right)dx}_{\text{a dot product}} = \int_0^L\left(\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{nm}}$$

$$= \sum_{n=1}^{\infty} a_n\left(\frac{L}{2}\delta_{nm}\right)$$

kills all terms in the sum except $n=m$

$$= a_m \frac{L}{2}$$

$$\therefore \boxed{a_m = \left(\frac{2}{L}\right)\int_0^L y(x)\sin\left(\frac{m\pi x}{L}\right)dx}$$ Fourier's Trick

expansion coefficient

normalization

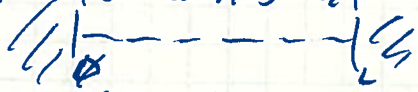
initial conditions

# Generalized Fourier Series

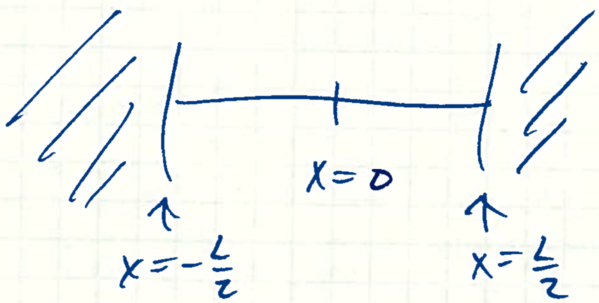
We've been studying a special type of Fourier Series called a "Fourier Sine Series"

$$y(x, t=\emptyset) = \sum_{n=1}^{\infty} a \sin\left(\frac{n\pi x}{L}\right)$$

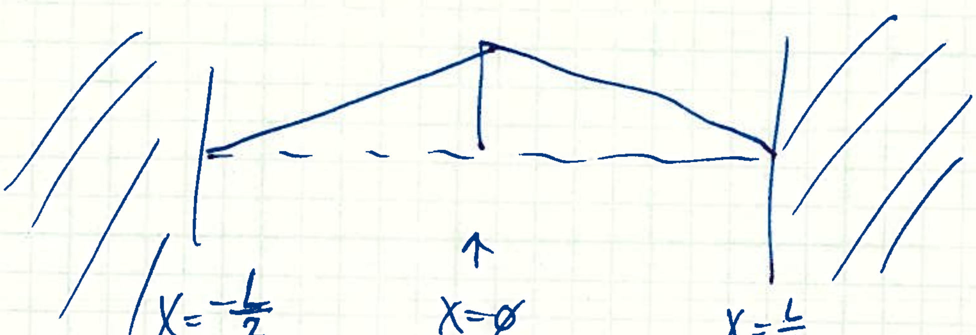
We like this series because it describes a string attached to walls at $x=\emptyset$ & $x=L$.




In general, however, we may choose to attach our string at other locations, like:



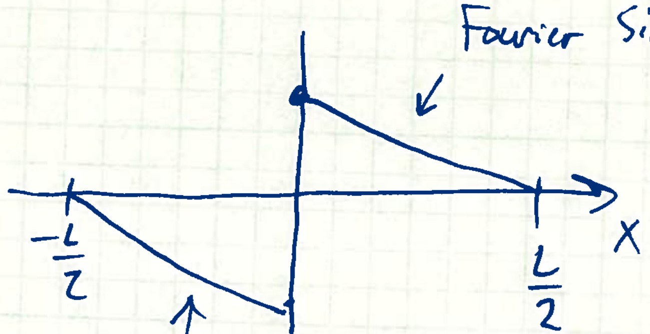

Suppose, for example, that the initial shape of our string is a triangle, and our coordinate system is centered on the middle of the string:



Can we represent this shape as a sum of sine functions?

Answer: No, because sine functions are odd and this function is even. If we tried, we would get

Fourier Sine Series is OK here




but it gives the wrong sign here

It's because every term in the series is odd:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore f(-x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi(-x)}{L}\right) = -\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= -f(x) \leftarrow \text{odd function}$$

To represent an even function, we'll need a "Fourier Cosine Series"

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \leftarrow \text{"Fourier Cosine Series" for even functions.}$$

How do we determine the expansion coefficients $\{a_n\}$ for this series?

Answer: The cosine functions are orthogonal:

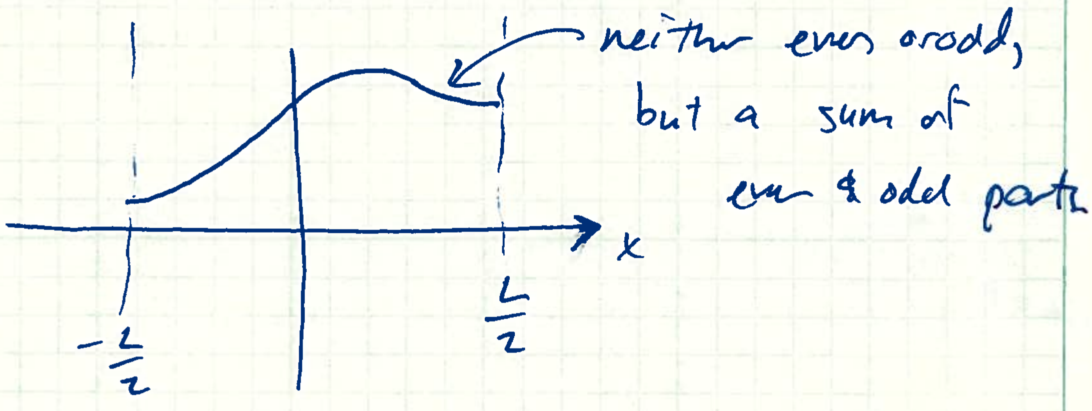
$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{nm}$$

Therefore Fourier's Trick works for them also:

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

In general, an arbitrary function is neither even or odd, but is a sum of even and odd parts:

$$f(x) = f_{odd}(x) + f_{even}(x)$$

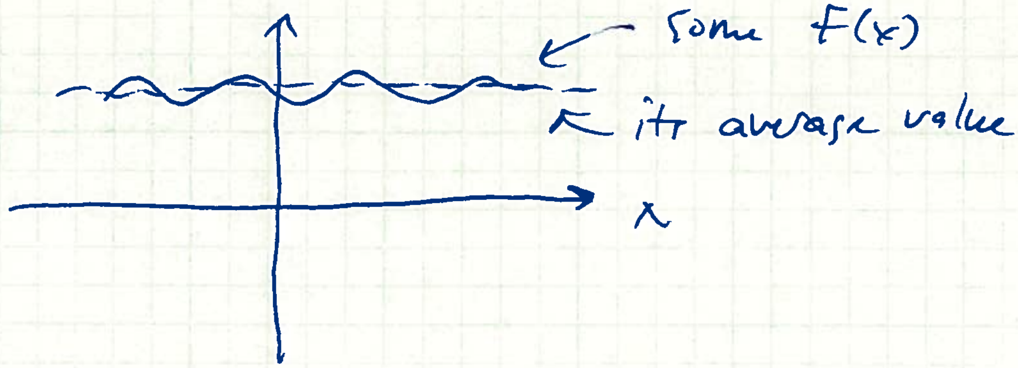



To represent a function like this, we need both Sine & Cosine Terms:

$$f(x) = \sum_n \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

But there's still one thing missing:

If the average value of the function is zero, then sines & cosines are fine. But if the function has a y-offset, then we have to add a constant:

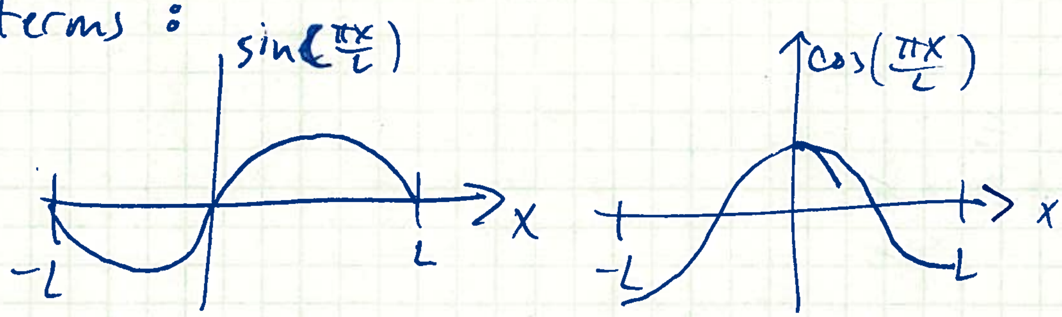



Finally, a complete, general Fourier Series is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_n \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]}$$

average value of $f(x)$

This series can represent almost any periodic function. But what is the period? Look at the $n=1$ terms:



The full period is $\boxed{2L}$.

Generalized Fourier Series:

$f(x)$ is ① periodic with period $2L$

② "square integrable" from $-L$ to $L$.

Then $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Note that normalization factor has changed because now we integrate over a distance of $2L$ instead of $L$.

Also, note that

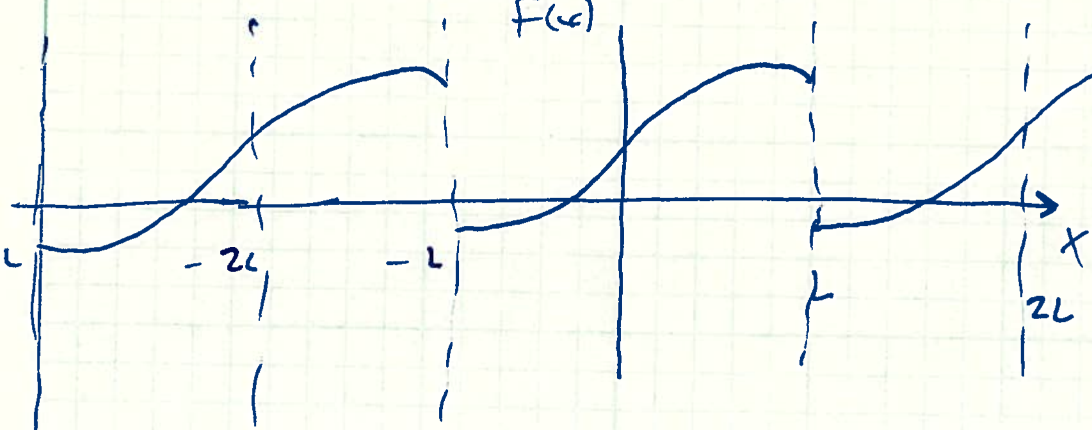
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{(0)\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

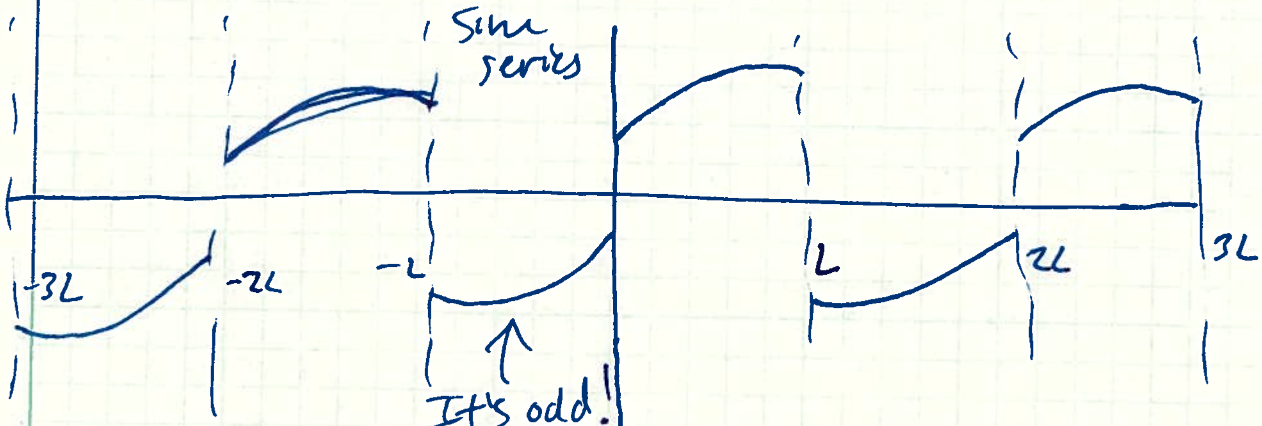
$= 2 \times$ average value of $f(x)$ between $-L$ & $L$

Picture it:

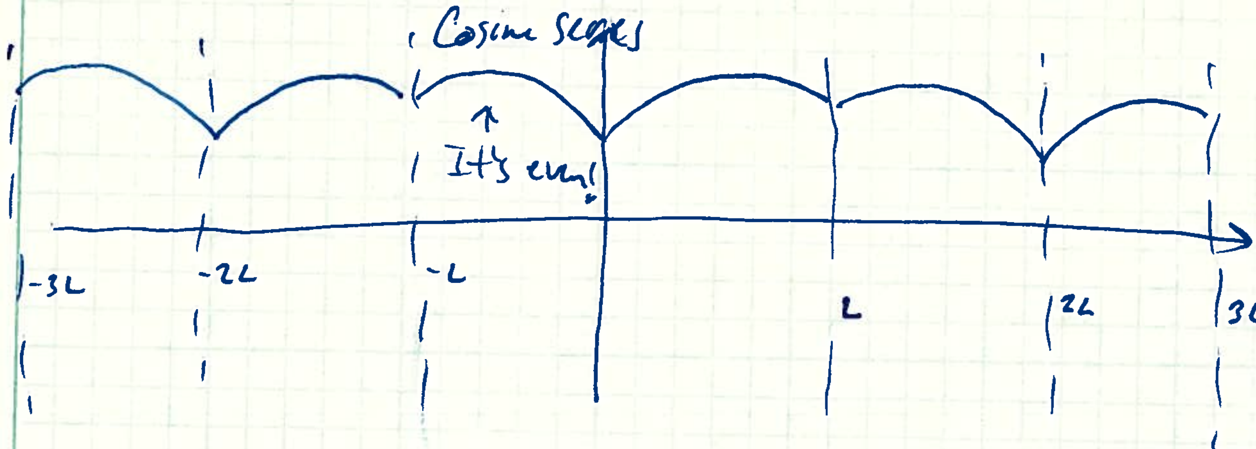
Suppose $F(x)$ looks like this:



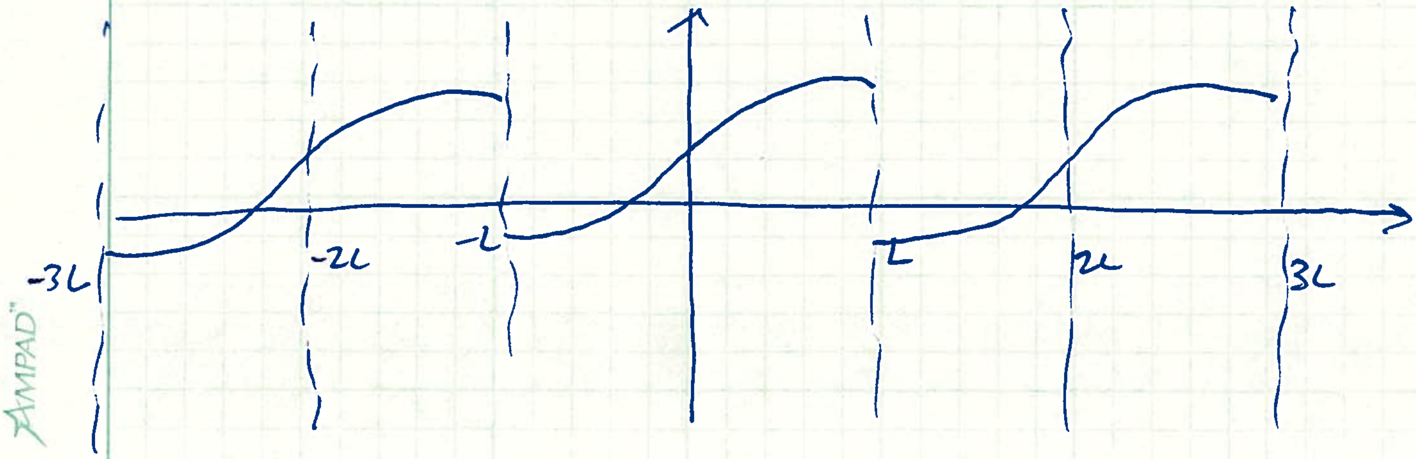

The sine series looks like this:




The cosine series looks like this

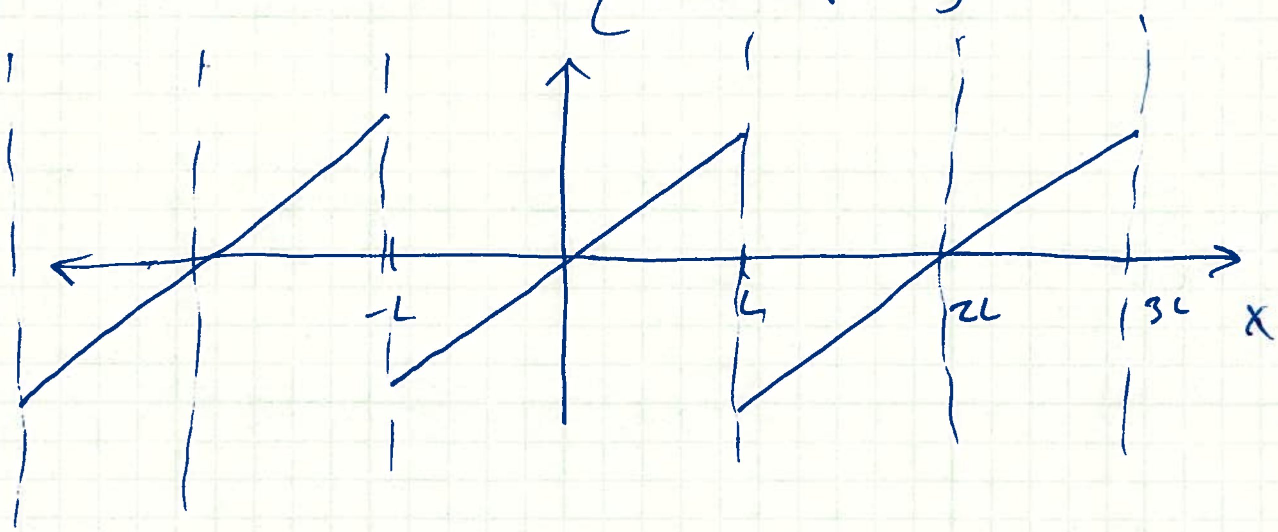


The complete, generalized Fourier Series looks exactly like $f(x)$




Example Calculation of a complete Fourier Series

**Sawtooth:** $f(x) = \begin{cases} x, \text{ for } -L < x < L \\ \text{and repeating} \end{cases}$




Fourier coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} x\,dx = \emptyset \leftarrow \text{average value is zero}$$

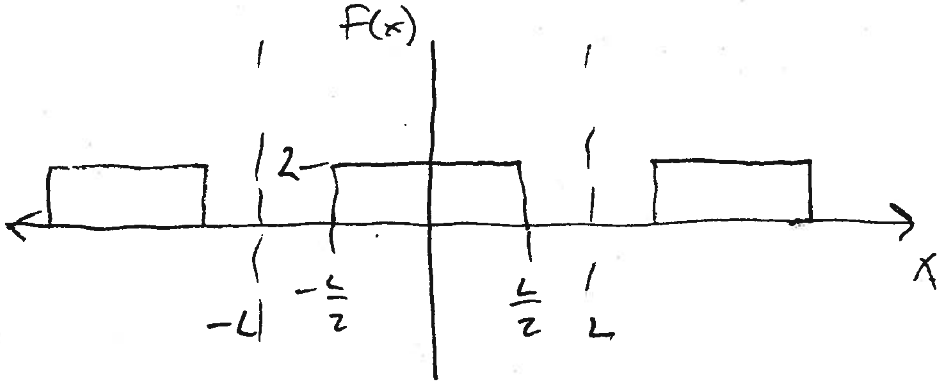
$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{x \cos\left(\frac{n\pi x}{L}\right)}_{\text{integrand is odd}} dx = \emptyset \leftarrow \text{no cosine terms! (Function is odd)}$$

$$b_n = \frac{1}{L}\underbrace{\int_{-L}^{L} x \sin\left(\frac{n\pi x}{L}\right) dx}_{\text{look up this integral or integrate by parts}} = -2\left(\frac{L}{n\pi}\right)\cos(n\pi) + 2\left(\frac{L}{n\pi}\right)^2 \cancelto{\emptyset}{\sin(n\pi)}$$

$$\therefore \quad \boxed{b_n = \frac{2L}{n\pi}(-1)^{n+1}}$$

$$\therefore \quad f(x) = \frac{2L}{\pi}\sin\left(\frac{\pi x}{L}\right) - \frac{2L}{2\pi}\sin\left(\frac{2\pi x}{L}\right) + \frac{2L}{3\pi}\sin\left(\frac{3\pi x}{L}\right) + \ldots$$

Example: Another type of square wave:




$$f(x) = \begin{cases} 0, & -L < x < -\frac{L}{2} \\ 2, & -\frac{L}{2} < x < \frac{L}{2} \\ 0, & \frac{L}{2} < x < L \end{cases} \quad \text{\& repeating with period } 2L.$$

It's an even function of x, so ~~only~~ the sine terms will be zero:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \emptyset$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2\,dx = 2 = 2 \times \text{average value of } f(x)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 2\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \left(\frac{2}{L}\right)\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \left(\frac{2}{n\pi}\right)\left(\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right) = \left(\frac{4}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right) = \begin{cases} \emptyset, & n \text{ even} \\ \frac{4}{n\pi}(-1)^{(n-1)/2}, & n \text{ odd} \end{cases}$$

$$a_n = \left(\frac{4}{n\pi}\right)(-1)^{(n-1)/2} \quad \text{for } n \text{ odd only}$$

$$\therefore \quad f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + a_3 \cos\left(\frac{3\pi x}{L}\right) + \ldots$$

$$\frac{a_0}{2} \to 1, \quad a_1 \to \frac{4}{\pi}, \quad a_2 \to \emptyset, \quad a_3 \to -\frac{4}{3\pi}$$

$$= 1 + \frac{4}{\pi}\cos\left(\frac{\pi x}{L}\right) - \frac{4}{3\pi}\cos\left(\frac{3\pi x}{L}\right) + \ldots$$

$$= 1 + \sum_{n=\text{odd}}^{\infty} (-1)^{(n-1)/2} \left(\frac{4}{n\pi}\right) \cos\left(\frac{n\pi x}{L}\right)$$

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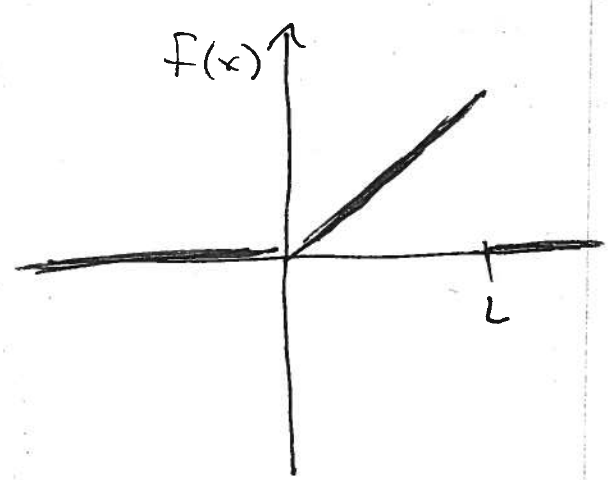
Consider this function

$$f(x) = \begin{cases} \emptyset, & x < \emptyset \\ x, & 0 \le x \le L \\ \emptyset, & x > L \end{cases}$$

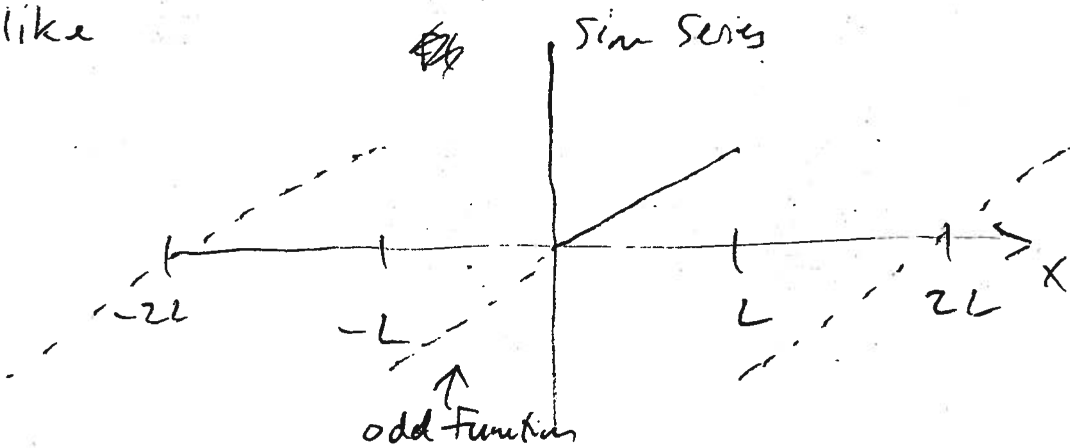



Suppose we wish to represent this function as a Fourier Series in the interval $[0, L]$, and we don't care if the Fourier series gives $\emptyset$ outside the interval. So we would like to write

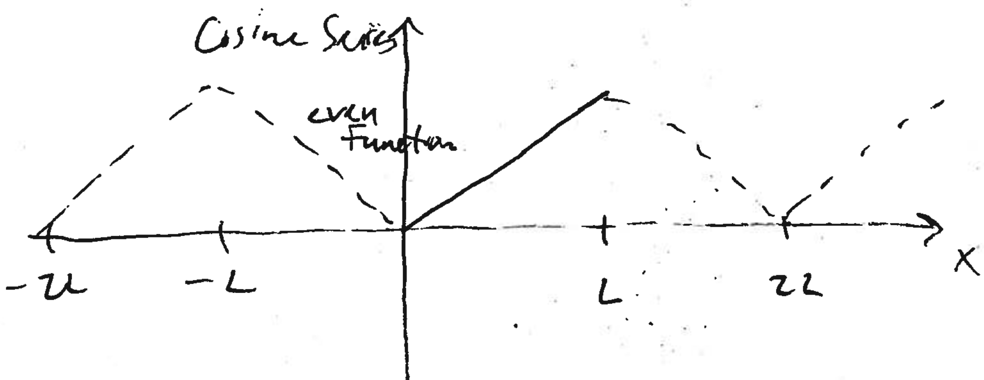
$$f(x) = \begin{cases} \emptyset, & x < \emptyset \\ \text{A Fourier Series}, & 0 \le x \le L \\ \emptyset, & x > L \end{cases}$$

Will we need sin terms, cosine terms, or both?

Answer: we can use either a Sine Series or a Cosine series. The Sine series will look like




The Cosine Series will look like




Between $x=\emptyset$ and $x=L$, either one can represent $f(x)$.

Sine Series Representation:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right) - \left(\frac{L}{n\pi}\right) x \cos\left(\frac{n\pi x}{L}\right)\right|$$

$$= \frac{2}{L}\left[-\left(\frac{L^2}{n\pi}\right)\underbrace{\cos(n\pi)}_{(-1)^n}\right] = \boxed{\frac{2L}{(n\pi)}(-1)^{n+1}}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2L}{(n\pi)}\sin\left(\frac{n\pi x}{L}\right)}$$

Cosine Series Representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L}\int_0^L x\,dx = \left(\frac{2}{L}\right)\left(\frac{1}{2}L^2\right) = L = 2 \times \text{average value of } f(x)$$

$$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)x\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$= \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos(n\pi) - \left(\frac{L^2}{n\pi}\right)\right]$$

$$= \frac{2L}{(n\pi)^2}\underbrace{\left(\cos(n\pi) - 1\right)}_{-2,\ 0,\ -2,\ 0,\ \dots}$$

$$\boxed{= \frac{-4L}{(n\pi)^2}}, \quad n = \text{odd only.}$$

$$\boxed{f(x) = \frac{L}{2} + \sum_{n=\text{odd}}^{\infty} \frac{-4L}{(n\pi)^2} \times \cos\left(\frac{n\pi x}{L}\right)}$$

So mathematically we can represent our function as either a Sine Series or Cosine Series, as long as we only care about the result between $x = 0$ and $x = L$.

But which series is physically relevant for a string connected to walls at $x=0$ and $x=L$?


Answer: For this physical system, only the sine series is relevant, because the sin functions are the normal modes for this system. That means we can write the time evolution in a trivial way for the sine series:

$$\boxed{y(x,t) = \sum_{n=} (-1)^{n+1} \frac{2L}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}$$ ✓ correct

The cosine series is of no use to use for this system because the cosine terms do not satisfy the equation of motion and boundary condition:

$$y(x,t) \neq \boxed{\cancel{\frac{L}{2} + \sum_{n=\text{odd}}^{\infty} \frac{4L}{(n\pi)^2} (-1)^{(n+1)/2} \cos\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}}}$$

Wrong! Cosine terms are not normal modes!