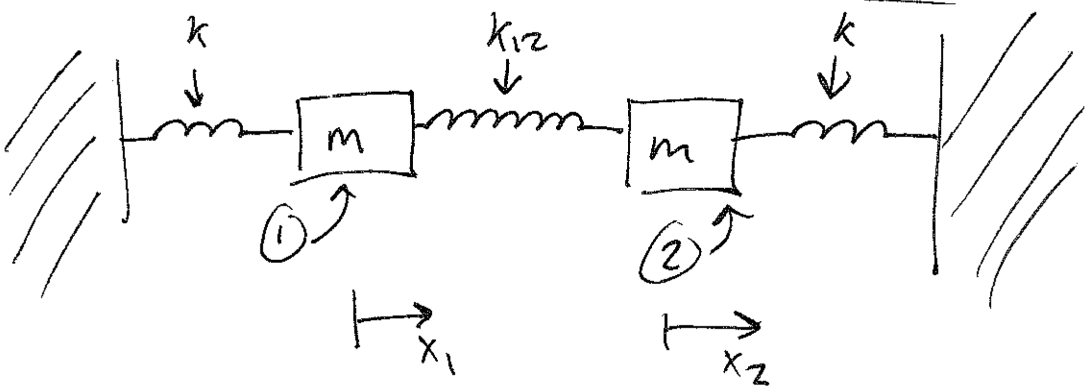
Two Coupled Mechanical Oscillators



3 springs, but just 2 spring constants.

x_1: displacement of 1 from equilibrium
x_2: " 2 "

Newton's 2nd Law

Force on 1: $F_1 = -kx_1 - k_{12}(x_1 - x_2) = m\ddot{x}_1$
Force on 2: $F_2 = -kx_2 + k_{12}(x_1 - x_2) = m\ddot{x}_2$

Equations of Motion:

$$\boxed{\begin{aligned} m\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 &= 0 \\ m\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 &= 0 \end{aligned}}$$

Coupled Differential Equations.

x_1 and x_2 appear in both equations.
we must solve for $x_1(t)$ & $x_2(t)$ ~~at the~~ simultaneously.

~~Math's~~ Our strategy: Let's look for solutions where both masses execute harmonic motion at the same frequency.

Guessed Solution:

$$X_1 = A_1 e^{i(\omega t + \delta_1)}$$
$$X_2 = A_2 e^{i(\omega t + \delta_2)}$$

ω is unknown. But the key feature of our guess is that both X_1 AND X_2 oscillate at the same frequency.

Normal Mode * If all the masses in a system oscillate at the same frequency, then we call the motion a "Normal Mode".

Normal Frequency: The frequency of a normal mode.

Let's Simplify Notation: re-write the guessed solution:

$$X_1 = A_1 e^{i(\omega t + \delta_1)} = \underbrace{\left(A_1 e^{i\delta_1}\right)}_{\equiv B_1} e^{i\omega t} \equiv B_1 e^{i\omega t}$$

B_1 is complex, it has two free parameters. (real & imaginary parts).

Also: $X_2 = B_2 e^{i\omega t}$

complex, 2 free parameters

Substitute the guessed solution into the equation of motion:

$$\boxed{\begin{aligned} m(-\omega^2 B_1) + (k+k_{12})B_1 - k_{12}B_2 &= 0 \\ m(-\omega^2 B_2) + (k+k_{12})B_2 - k_{12}B_1 &= 0 \end{aligned}}$$

$e^{i\omega t}$ has cancelled everywhere.

Gather Terms:

$$\boxed{\begin{aligned} (k+k_{12}-m\omega^2)B_1 - k_{12}B_2 &= 0 \quad \text{Eq. 1} \\ -k_{12}B_1 + (k+k_{12}-m\omega^2)B_2 &= 0 \quad \text{Eq. 2} \end{aligned}}$$

To have a solution, the determinant must equal zero:

$$(k+k_{12}-m\omega^2)^2 - k_{12}^2 = 0$$

$$k+k_{12}-m\omega^2 = \pm k_{12}$$ 2 possibilities.

$$\boxed{\omega = \sqrt{\frac{k+k_{12}\pm k_{12}}{m}}}$$

We have found $\underline{2}$ normal mode frequencies:

Let's call them:

$$\boxed{\begin{aligned} \omega_s &= \text{smaller frequency} = \sqrt{\frac{k}{m}} \\ \omega_L &= \text{larger frequency} = \sqrt{\frac{k+2k_{12}}{m}} \end{aligned}}$$

So the small frequency solution is:

$$x_1 = B_1 e^{i\omega_s t}$$
$$x_2 = B_2 e^{i\omega_s t} \quad , \quad \omega_s = \sqrt{\frac{k}{m}}$$

But we are not done. We can show that for this solution, we must have $B_1 = B_2$. To see this, substitute $\omega_s = \sqrt{k/m}$ into Eq. 1 & Eq. 2:

$$\boxed{\begin{aligned} (k + k_{12} - k) B_1 - k_{12} B_2 &= \emptyset \\ -k_{12} B_1 + (k + k_{12} - k) B_2 &= \emptyset \end{aligned}}$$

or

$$\boxed{\begin{aligned} k_{12} (B_1 - B_2) &= \emptyset \\ -k_{12} (B_1 - B_2) &= \emptyset \end{aligned}}$$

or $\boxed{B_1 = B_2}$ For the small frequency solution.

Let's call it $B_1 = B_2 \equiv \underline{B_s}$ = "small frequency case.

Then our solution is

$$\boxed{\begin{aligned} X_1 &= B_s e^{i\omega_s t} \\ X_2 &= B_s e^{i\omega_s t} \end{aligned}}$$

B_s = complex = 2 free parameters, (real & imaginary parts)

This is called the "symmetric mode", because both oscillators have exactly the same motion. ⇒ Amplitude, phase, and frequency are identical.

<u>Large Frequency mode</u>: Exactly the same methods leads to

$$B_1 = -B_2 \quad \text{for } \omega_L = \sqrt{\frac{k+2k_{12}}{m}}$$

(−) sign means
That x_1 & x_2 are
out-of-phase by $180°$.

Call $B_1 = B_L$. Then the large frequency solution is

$$\boxed{\begin{aligned} x_1 &= B_L e^{i\omega_L t} \\ x_2 &= -B_L e^{i\omega_L t} \end{aligned}}$$

} B_L has 2 free parameters: real & imaginary parts.

We call this the "anti-symmetric mode" because the two oscillators are $180°$ out-of-phase with each other.

General Solution

The 2 normal mode solution are the simplest type of motion that the system may execute. But we can find a general solution by adding the normal mode solutions. This works because the equations of motion are linear.

$$\boxed{\begin{aligned} x_1 &= B_s e^{i\omega_s t} + B_L e^{i\omega_L t} \\ x_2 &= B_s e^{i\omega_s t} - B_L e^{i\omega_L t} \end{aligned}}$$

The most general solution.

Note that we have 4 free parameters: the real & imaginary parts of B_s & B_L. We need 4 initial conditions to specify them:

position and velocity of (1) at $t=0$
&
(2) at $t=0$.

Let's take the real part and apply one particular set of initial conditions:

$$x_1 = b_s \cos(\omega_s t + \delta_s) + b_L \cos(\omega_L t + \delta_L)$$

$$x_2 = b_s \cos(\omega_s t + \delta_s) - b_L \cos(\omega_L t + \delta_L)$$

$b_s, \delta_s, b_L, \delta_L$ are free parameters.

Suppose that

$$x_1(t=0) = a$$
$$\dot{x}_1(t=0) = 0$$
$$x_2(t=0) = 0$$
$$\dot{x}_2(t=0) = 0$$

Then the $\dot{x}_1$ & $\dot{x}_2$ requirements are:

$$\boxed{\begin{aligned} \dot{x}_1 &= -\omega_s b_s \sin(\delta_s) - \omega_L b_L \sin(\delta_L) = 0 \\ \dot{x}_2 &= -\omega_s b_s \sin(\delta_s) + \omega_L b_L \sin(\delta_L) = 0 \end{aligned}}$$

Add these equations: $-2\omega_s b_s \sin(\delta_s) = 0 \Rightarrow \boxed{\delta_s = 0}$

Subtract these equations: $-2\omega_L b_L \sin(\delta_L) = 0 \Rightarrow \boxed{\delta_L = 0}$

And the $x_1 = a$ and $x_2 = 0$ requirements are

$$x_1 = b_s \cos(\underset{0}{\delta_s}) + b_L \cos(\underset{0}{\delta_L}) = a$$

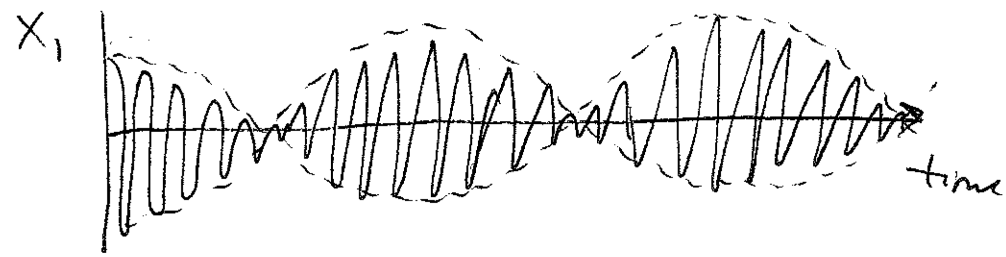
$$x_2 = b_s \cos(\delta_s) - b_L \cos(\delta_L) = 0$$

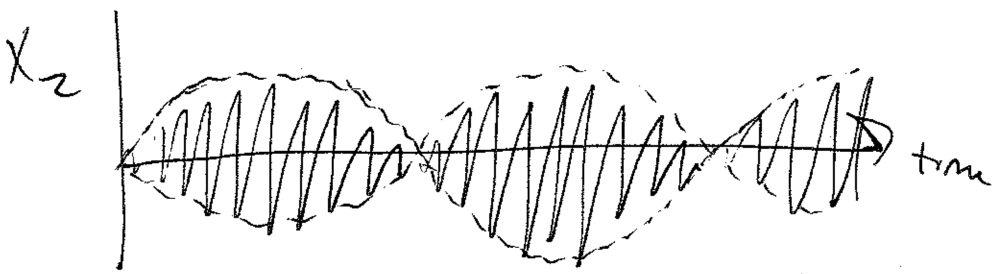
or $\boxed{\begin{aligned} b_s + b_L &= a \\ b_s - b_L &= 0 \end{aligned}} \Longrightarrow \boxed{\begin{aligned} b_s &= \frac{a}{2} \\ b_L &= \frac{a}{2} \end{aligned}}$

So the complete solution for these initial conditions is

$$\boxed{\begin{aligned} x_1 &= \frac{a}{2}\cos(\omega_s t) + \frac{a}{2}\cos(\omega_L t) \\ x_2 &= \frac{a}{2}\cos(\omega_s t) - \frac{a}{2}\cos(\omega_L t) \end{aligned}}$$

What does it look like?:





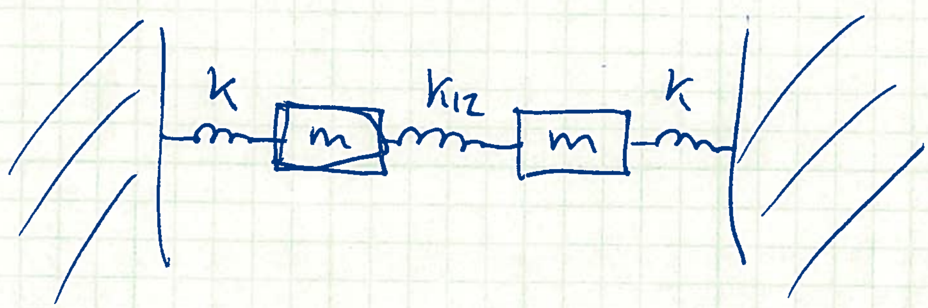
What is a normal mode?

A normal mode is a special type of motion for a multi-particle harmonic system. It's defining characteristic is that the time evolution is very simple: every particle oscillates at the same frequency

For a system to satisfy this condition, the amplitudes & phases of the various particles must be related to each other. So to find a normal mode, we must do 2 things:

1. determine which frequencies are normal-mode frequencies.
2. determine the relationship between the amplitudes of motion for ~~that~~ each normal mode

Example Coupled Mechanical Oscillator (2 particle system)



We solved the equation of motion and found 2 normal frequencies:

$$\omega_S = \sqrt{\frac{k}{m}} = \text{"small frequency"}$$

$$\omega_L = \sqrt{\frac{k+2k_{12}}{m}} = \text{"large frequency"}$$

We also found the amplitude relationship:

For $\underline{\omega_S}$:

$$\boxed{\begin{aligned} x_1 &= B_S e^{i\omega_S t} \\ &\updownarrow \text{ same amplitude and phase} \\ x_2 &= B_S e^{i\omega_S t} \end{aligned}}$$

For $\underline{\omega_L}$:

$$\boxed{\begin{aligned} x_1 &= B_L e^{i\omega_L t} \\ &\updownarrow \text{ same amplitude, opposite phase} \\ x_2 &= - B_L e^{i\omega_L t} \end{aligned}}$$

opposite phase

(Recall that B_S & B_L are complex, so the phase at $t=0$ is absorbed into B_S & B_L.)

Notice that our 2-particle system has 2 normal modes. In <u>general, an N-particle</u> <u>system will have N-normal modes.</u>

We determined B_L & B_S for a particular set of initial conditions:

$$x_1(t=0) = a \quad , \quad \dot{x}_1(t=0) = 0$$
$$x_2(t=0) = 0 \quad , \quad \dot{x}_2(t=0) = 0$$

Complete solution: (Real part)

$$x_1(t) = \frac{a}{2}\cos(\omega_s t) + \frac{a}{2}\cos(\omega_L t)$$

$$x_2(t) = \frac{a}{2}\cos(\omega_s t) - \frac{a}{2}\cos(\omega_L t)$$

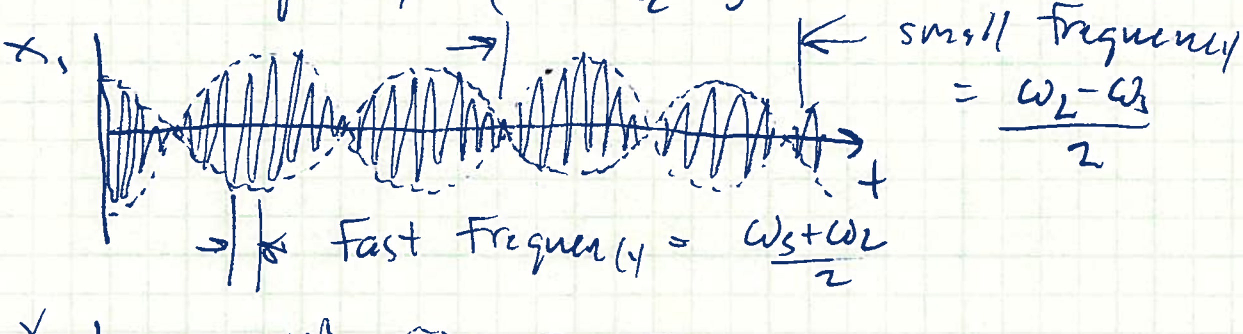
Question: What does this solution look like?

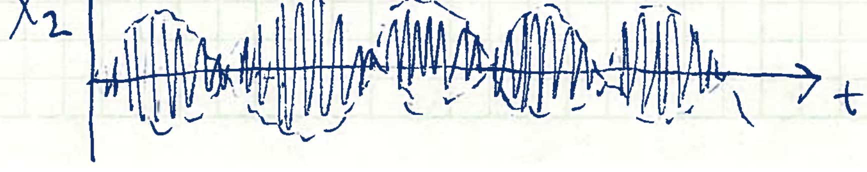
Answer: Re-write it using a (dreaded) trig identity:

$$x_1(t) = a\cos\left(\frac{(\omega_L - \omega_s)}{2}t\right)\cos\left(\left(\frac{\omega_L+\omega_s}{2}\right)t\right)$$

$$x_2(t) = a\sin\left(\frac{(\omega_L - \omega_s)}{2}t\right)\sin\left(\left(\frac{\omega_L+\omega_s}{2}\right)t\right)$$

So we have two harmonic functions multiplied together. There is a "fast oscillation" whose frequency is the average of ω_L & ω_s. But the amplitude goes up and down with a slow frequency $\left(\frac{(\omega_L - \omega_s)}{2}\right)$.





Question

Why do we care about normal modes?

Answer: Two reasons

① The general solution, valid for any initial conditions, can be written as a sum over normal modes:

$$\left.\begin{aligned} x_1 &= B_S e^{i\omega_S t} + B_L e^{i\omega_L t} \\ x_2 &= B_S e^{i\omega_S t} - B_L e^{i\omega_L t} \end{aligned}\right\} \text{ sum over normal mode solutions}$$

Any valid motion of the system can be described by specifying four initial conditions: Real & Imag. parts of B_S & real & imaginary parts of B_L.

② The time-evolution of each normal mode is extremely simple: simply multiply by $e^{i\omega_i t}$ for each mode.

This is easier to see if we simplify our notation. Let $\vec{X} \equiv (x_1, x_2)$ be a vector which describes the current position of m_1 & m_2.

Rename: $B_S = a_1$, $\omega_S = \omega_1$

$B_L = a_2$, $\omega_L = \omega_2$

Then

$$\boxed{\begin{aligned} x_1(t) &= a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} \\ x_2(t) &= a_1 e^{i\omega_1 t} - a_2 e^{i\omega_2 t} \end{aligned}}$$

With vector notation I can combine two equations into one:

$$\left(x_1(t),\ x_2(t)\right) = \underset{\text{initial conditions}}{a_1}\underbrace{(1,1)}_{\text{constant vector}}\overset{\text{time evolution}}{e^{i\omega_1 t}} + \underset{\text{initial conditions}}{a_2}\underbrace{(1,-1)}_{\text{constant vector}}e^{i\omega_2 t}$$

I can simplify the notation further if I define

$\vec{q}_1 \equiv$ constant vector $\equiv (1,1)$

$\vec{q}_2 \equiv$ constant vector $\equiv (1,-1)$

Then
$$\boxed{\vec{x}(t) = a_1\vec{q}_1 e^{i\omega_1 t} + a_2\vec{q}_2 e^{i\omega_2 t}}$$

Should we make it even simpler? Use summation notation:

$$\boxed{\vec{x}(t) = \sum_{n=1}^{2} a_n \vec{q}_n e^{i\omega_n t}}$$

This equation says exactly the same thing as our original general solution, but it is written more compactly and elegantly.

For example, we still need 4 initial conditions to specify $Re(a_1)$, $Im(a_1)$, $Re(a_2)$, $Im(a_2)$.

The vectors $\vec{q}_1$ & $\vec{q}_2$ are the "normal mode eigenvectors". They ~~a~~ are fixed, constant

vectors which describe the fixed relationship between the amplitudes of ~~x_1~~ x_1 & ~~x_2~~ x_2 in each normal mode.

$\vec{q}_1 = (1,1)$ = "x_1 & x_2 have the same amplitude and phase in this mode" (symmetric mode)

$\vec{q}_2 = (1,-1)$ = "x_1 & x_2 have the same amplitude, but a phase difference of 180°, in this mode" (antisymmetric mode)

In general, the system is ~~[illegible]~~ not in a single normal mode, but is in a sum, or superposition, of normal modes. The fixed relationship between amplitudes of x_1 & x_2 will only occur when the system happens to be in a pure normal mode.

~~The amp~~

a_1 & a_2, which are determined by the initial conditions, are called the "normal coordinates". They describe "how much of each normal mode" is in the ~~[illegible]~~ motion.

If we want to simplify further we could absorb the time evolution factor into a_1 & a_2:

$$\vec{x}(t) = \sum_{n=1}^{2} a_n \vec{q}_n e^{i\omega_n t} = \sum_{n=1}^{2} \underbrace{\left(a_n e^{i\omega_n t}\right)}_{a_n(t) \equiv a_n e^{i\omega_n t}} \vec{q}_n$$

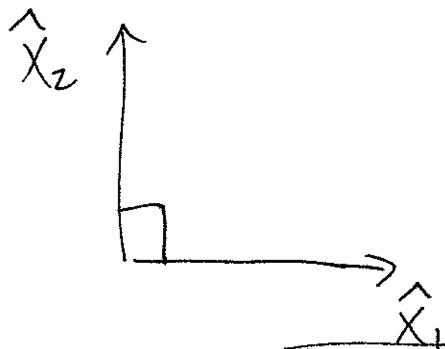
$$= \sum_{n=1}^{2} a_n(t)\vec{q}_n = a_1(t)\vec{q}_1 + a_2(t)\vec{q}_2$$

Since $a_n(t) \equiv a_n e^{i\omega_n t}$, we see that each normal mode evolves in time by picking up a phase factor of $e^{i\omega_n t}$. Notice that the magnitude of each normal mode component does not change, only its phase:

$$|a_n(t)| = |a_n e^{i\omega_n t}| = |a_n| \underbrace{|e^{i\omega_n t}|}_{\text{magnitude} = 1} = |a_n| = \text{constant}.$$

Therefore, whatever normal modes we have at $t = \emptyset$, we will have forever. Normal modes do not appear or dissappear as time goes forward. (This is because we assumed no drag forces and no driving forces either. Drag forces would cause the amplitudes to decay, driving forces would cause them to grow (transient effect).)

Consider 2 dimensional unit vectors which are perpindicular or orthogonal to each other:



Then $\hat{X}_1 \cdot \hat{X}_1 = 1$

$\hat{X}_2 \cdot \hat{X}_2 = 1$

$\hat{X}_1 \cdot \hat{X}_2 = \emptyset$

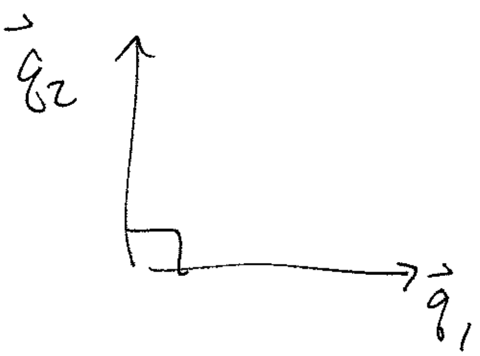
Summarizing, $\boxed{\hat{X}_i \cdot \hat{X}_j = \delta_{ij},}$

where δ_{ij} = "Kronecker Delta"

$$= \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

The Kronecker Delta describes the orthogonality of these unit vectors.

Suppose we have 2 vectors which are orthogonal but not of unit length:



Then $\vec{q}_1 \cdot \vec{q}_1 = |\vec{q}_1|^2$

$\vec{q}_2 \cdot \vec{q}_2 = |\vec{q}_2|^2$

$\vec{q}_1 \cdot \vec{q}_2 = \emptyset$

Summarizing, $\boxed{\vec{q}_i \cdot \vec{q}_j = |\vec{q}_i|^2 \delta_{ij} = |\vec{q}_j|^2 \delta_{ij}}$

This is true for the eigenvectors of the coupled oscillator system:

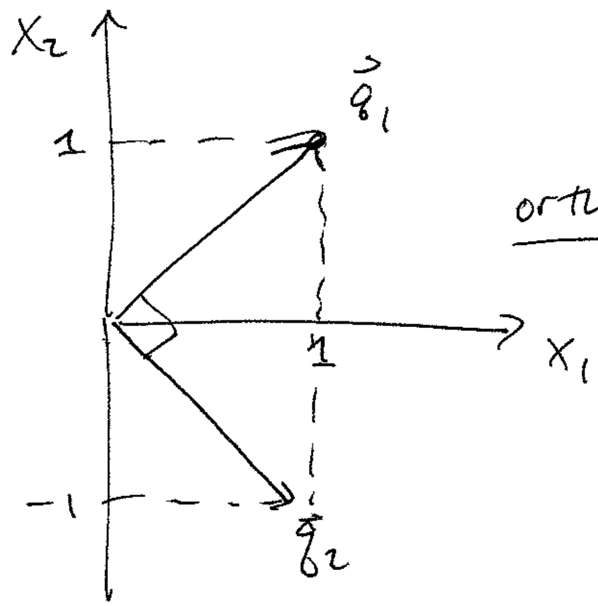
symmetric mode $= \vec{q}_1 = (1, 1)$

anti-symmetric mode $= \vec{q}_2 = (1, -1)$

$$\vec{q}_1 \cdot \vec{q}_2 = (1, 1) \cdot (1, -1) = 1 - 1 = \emptyset$$

These eigenvectors are <u>orthogonal</u>.

Geometrically,



<u>orthogonal</u>.

We can use the orthogonality of the eigenvectors to quickly incorporate the initial conditions into the general solution.

Recall the general solution of the 2-coupled oscillator system:

$$\vec{x} = \sum_{n=1}^{2} c_n \vec{q}_n e^{i\omega_n t}$$

← time evolution.

c_n: initial conditions

$\vec{q}_n$: eigenvectors of the normal modes

Remember, the expansion coefficients $\{c_n\}$ are complex:

$$c_n \equiv a_n + i b_n$$

So we have 4 free parameters to determine using the 4 initial conditions:

$$\left.\begin{aligned} c_1 &= a_1 + i b_1 \\ c_2 &= a_2 + i b_2 \end{aligned}\right\} \boxed{a_1, b_1, a_2, b_2}$$

Initial Conditions: $\boxed{\begin{matrix} x_1(t=\emptyset), & x_2(t=\emptyset) \\ v_1(t=\emptyset), & v_2(t=\emptyset) \end{matrix}}$

Question: How can we calculate the $\{c_n\}$ for our particular set of initial conditions?

Answer: Use the orthogonality of the eigenvectors. First put the initial conditions into vector form:

$$\text{Let } \vec{X}_0 \equiv (x_1(t=\emptyset), x_2(t=\emptyset))$$

$$\text{and let } \vec{V}_0 \equiv (v_1(t=\emptyset), v_2(t=\emptyset))$$

$$= (\dot{x}_1(t=\emptyset), \dot{x}_2(t=\emptyset))$$

The Real part of the general solution must give $\vec{x}_0$ at $t=\emptyset$:

$$\vec{x}_0 = \mathrm{Re}\left[\sum_{n=1}^{2} c_n \vec{q}_n \underbrace{e^{i\omega_n(\emptyset)}}_{1}\right] \quad \leftarrow t=\emptyset.$$

$$= \mathrm{Re}\left[\sum_{n=1}^{2} (a_n + i b_n)\, \vec{q}_n\right]$$

$$\boxed{\vec{x}_0 = \sum_{n=1}^{2} a_n \vec{q}_n}$$

Now consider this dot product:

$$\vec{x}_0 \cdot \vec{q}_1 = \left(\sum_{n=1}^{2} a_n \vec{q}_n\right) \cdot \vec{q}_1$$

$$= \sum_{n=1}^{2} a_n \underbrace{(\vec{q}_n \cdot \vec{q}_1)}$$

orthogonal, so $\vec{q}_n \cdot \vec{q}_1 = |\vec{q}_n||\vec{q}_1|\,\delta_{n1}$

$$= \sum_{n=1}^{2} a_n |\vec{q}_n||\vec{q}_1| \underbrace{\delta_{n1}}$$

Kronecker Delta kills all the terms in the sum <u>except</u> the $n=1$ term:

$$= a_1 |\vec{q}_1||\vec{q}_1|$$

$$= a_1 |\vec{q}_1|^2$$

Summarizing: $\boxed{a_1 = \dfrac{\vec{X}_0 \cdot \vec{q}_1}{|\vec{q}_1|^2}}$

We can calculate a_1 by taking the dot product of $\vec{X}_0 = (x_1(t=\emptyset), x_2(t=\emptyset))$ with $\vec{q}_1$ and dividing by $|\vec{q}_1|^2$.

Similarly, we can show that

$$a_2 = \frac{\vec{X}_0 \cdot \vec{q}_2}{|\vec{q}_2|^2}$$

In general, $\boxed{a_n = \dfrac{\vec{X}_0 \cdot \vec{q}_n}{|\vec{q}_n|^2}}$ This is how we calculate the $\{a_n\}$.

We call this "Fourier's Trick".

We can use a similar trick to calculate the $\{b_n\}$ from the initial velocities:

General Solution: $\vec{X} = \mathrm{Re}\left[\sum_{n=1}^{2} c_n \vec{q}_n e^{i\omega_n t}\right]$

so $\dot{\vec{X}} = \vec{V} = \mathrm{Re}\left[\sum_{n=1}^{2} i\omega_n c_n \vec{q}_n e^{i\omega_n t}\right]$

At $t=\emptyset$, $\vec{V}_0 = \mathrm{Re}\left[\sum_{n=1}^{2} i\omega_n c_n \vec{q}_n\right]$

$$\vec{V}_0 = \mathrm{Re}\left[\sum_{n=1}^{2} i\omega_n (a_n + i b_n)\vec{q}_n\right]$$

$$\vec{V}_0 = \sum_{n=1}^{2} -\omega_n b_n \vec{q}_n$$

Now consider this dot product:

$$\vec{V}_0 \cdot \vec{q}_1 = \left(\sum_{n=1}^{2} \left(-\omega_n b_n \vec{q}_n\right)\right) \cdot \vec{q}_1$$

$$= \sum_{n=1}^{2} -\omega_n b_n \underbrace{(\vec{q}_n \cdot \vec{q}_1)}$$

orthogonal, so
$\vec{q}_n \cdot \vec{q}_1 = |\vec{q}_n||\vec{q}_1| \, \delta_{n1}$

$$= \sum_{n=1}^{2} -\omega_n b_n |\vec{q}_n||\vec{q}_1| \underbrace{\delta_{n1}}$$

Kronecker Delta
kills all terms in
the sum except the
$n = 1$ term:

$$= -\omega_1 b_1 |\vec{q}_1||\vec{q}_1|$$

$$= -\omega_1 b_1 |\vec{q}_1|^2$$

Summarizing: $$b_1 = \frac{-\vec{V}_0 \cdot \vec{q}_1}{\omega_1 |\vec{q}_1|^2}$$

For b_2 we would find: $$b_2 = \frac{-\vec{V}_0 \cdot \vec{q}_2}{\omega_2 |\vec{q}_2|^2}$$

In general,

$$\boxed{b_n = \frac{-\vec{v}_0 \cdot \vec{q}_n}{\omega_n |\vec{q}_n|^2}}$$

Bottom Line: To calculate the coefficients $\{c_n\}$ from the initial conditions, take these dot products:

$$\boxed{a_n = \frac{\vec{x}_0 \cdot \vec{q}_n}{|\vec{q}_n|^2}} \quad \text{and} \quad \boxed{b_n = \frac{-\vec{v}_0 \cdot \vec{q}_n}{\omega_n |\vec{q}_n|^2}}$$

where $c_n \equiv a_n + i b_n$.

Example: 2 coupled oscillator.

Eigenvectors of the normal modes are:

$$\vec{q}_1 = (1, 1) \quad \text{and} \quad \vec{q}_2 = (1, -1).$$

Suppose our initial conditions are:

$$x_1(t=0) = a \qquad x_2(t=0) = 0$$
$$\dot{x}_1(t=0) = 0 \qquad \dot{x}_2(t=0) = 0.$$

Then $\vec{x}_0 = (a, 0)$

and $\vec{v}_0 = (0, 0)$.

Then the coefficients a_n & b_n are:

$$a_1 = \frac{\vec{x}_0 \cdot \vec{q}_1}{|\vec{q}_1|^2} = \frac{(a, \emptyset)\cdot(1, 1)}{(1^2 + 1^2)} = \frac{a}{2}$$

$$a_2 = \frac{\vec{x}_0 \cdot \vec{q}_2}{|\vec{q}_2|^2} = \frac{(a, \emptyset)\cdot(1, -1)}{1^2 + (-1)^2} = \frac{a}{2}$$

$$b_1 = \frac{-\vec{v}_0 \cdot \vec{q}_1}{\omega_1 |\vec{q}_1|^2} = \frac{-(\emptyset, \emptyset)\cdot(1, 1)}{\omega_1 (1^2 + 1^2)} = \emptyset$$

$$b_2 = \frac{-\vec{v}_0 \cdot \vec{q}_2}{\omega_2 |\vec{q}_2|^2} = \frac{-(\emptyset, \emptyset)\cdot(1, -1)}{\omega_2 (1^2 + (-1)^2)} = \emptyset$$

So $c_1 = \frac{a}{2}$

$c_2 = \frac{a}{2}$